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Effective string tension and renormalizability: string theory in a noncommutative space

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Abstract: We show that the one loop amplitudes of open and closed string theory in a constant background two-form tensor field are characterized by an effective string tension larger than the fundamental string tension, and by the appearance of antisymmetric and symmetric noncommutativity parameters. We derive the form of the phase functions normalizing planar and nonplanar tachyon scattering amplitudes in this background, verifying the decoupling of the closed string sector in the regime of infinite momentum transfer. We show that the functional dependence of the phase functions on the antisymmetric star product of external momenta permits interpretation as a finite wavefunction renormalization of vertex operators in the open string sector. Using world-sheet duality we clarify the regimes of finite and zero momentum transfer between boundaries, demonstrating the existence of poles in the nonplanar amplitude when the momentum transfer equals the mass of an on-shell closed string state. Neither noncommutativity parameter has any impact on the renormalizability of open and closed string theory in the wilsonian sense. We comment on the relationship to noncommutative scalar field theory and the UV-IR correspondence.

Keywords: Bosonic Strings, D-branes, Non-Commutative Geometry.

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Contents

1. Introduction

The idea that the coordinates of spacetime do not commute at sufficiently small distance scales has received new scrutiny with the discovery of nonperturbative backgrounds of String/M theory that are noncommutative spacetimes [1, 2]. The worldvolume of a Dbrane with constant background two-form tensor gauge fields is a simple and concrete example of a noncommutative spacetime in which live gauge and matter fields [3]–[8]. Motivated in part by the puzzling mix of ultraviolet and infrared effects recently observed in noncommutative scalar field theories [12], we will revisit the one-loop amplitudes of open and closed bosonic string theory in the presence of constant two-form background fields [3, 4].[1] We will attempt to "discover" possibly unusual properties of quantum field theory in this noncommutative space by deriving the field theory limit, in both open and closed string channels, of the one-loop tachyon scattering amplitude of bosonic string theory in a constant background field. The result is a surprise in many respects.

[1] Since this project was begun, many papers have appeared on this subject with overlapping results [14]–[19]. We will point out the differences at appropriate places in the text and in the conclusions.

In section 2, we point out that the "open string metric" [8] appearing in the B dependent normalization of the one-loop vacuum amplitude, originally computed in [3, 4], has a natural world-sheet interpretation as the effective string tension of open string theory in a constant background two-form tensor field. The effective string tension provides the natural scale with respect to which we measure momenta and energies in this theory. We will find that the effective string tension is always *larger* than the bare fundamental string tension, for nonvanishing background two-form fields. The scattering amplitudes of the theory are further characterized by the presence of antisymmetric and symmetric noncommutativity parameters which will be explored in sections 4 and 5. The noncommutativity scale is a priori distinct from the effective string tension and is associated with wavefunction renormalization for states in the open string sector.[2] We will find in section 3 that the noncommutativity parameters only enter the *finite* part of the Greens function on the annulus, thereby determining the vertex operator algebra and the external momentum dependent normalizations of the amplitudes, but with no bearing on the usual renormalizability properties of open and closed string theory in this background. The effective string tension, on the other hand, does appear in the short distance divergence of the Greens function for closely separated sources on the boundary of the world-sheet. This log divergence is absorbed in a renormalization of the bare open string coupling, precisely analogous to the case of free strings [13, 9].

The main distinction between our string theory interpretation and that of noncommutative scalar field theory is due to the application of open-closed string world-sheet duality. An analysis of the nonplanar amplitude in the open string channel indeed displays both the momentum dependent phase functions and the dependence on internal momentum transfer between boundaries, found in the corresponding nonplanar graphs of a noncommutative scalar field theory [12]. Momentum transfer between boundaries in string theory is addressed by using world-sheet duality to express the nonplanar amplitudes in the closed string channel. The potentially puzzling regime of zero momentum transfer is dominated by the exchange of a zero momentum massless state in the closed string sector. We will show that this limit is benign. The only renormalization necessary in our theory is the usual renormalization of the open string coupling, and the $\mathbf{p} \to 0$ limit of zero momentum transfer therefore commutes with taking the ultraviolet cut-off on momenta in the open string channel to infinity. It should be emphasized that in using world-sheet duality to rewrite the string amplitude in terms of the massless field theory limit of the closed string sector, we are outside the domain of the original noncommutative field theory which corresponds to the massless limit of the open string sector. Our conclusion is that open and closed string theory in a background two-form tensor field, albeit in a noncommu-

[2] A different notion of an effective string scale is described in [20], where the effective scale of string-like excitations in the IIB matrix model on a Von Neumann lattice is found to be identical to the noncommutativity scale of NCYM theory.

tative space, displays ordinary wilsonian behavior as regards renormalizability. The finite wavefunction renormalization of external states and the finite renormalization of the string tension are the main remnant signals of the noncommutative nature of the embedding spacetime.[3]

In what follows, we give a path integral derivation of both the one-loop vacuum amplitude and the planar, and nonplanar, one-loop tachyon scattering amplitudes in open and closed bosonic string theory in a constant background two-form tensor field. A path integral derivation of the N point closed string tachyon scattering amplitude on the torus for free strings was given in [13]. In sections 2 and 3, we adapt this calculation to open and closed string theory in a constant background two-form tensor field. The method enables us to derive the Greens function and the renormalized scattering amplitudes inclusive of normalization, and of all of the background field dependence. In order to make the paper self-contained, we include in the appendix a pedagogical discussion of the method in [13] with necessary extensions used by us to calculate the bulk Greens function on the annulus in constant background fields.

In section 4 we derive the precise form of the momentum dependence of the phase functions normalizing the planar one-loop amplitudes, assuming the simplifying kinematics of forward momentum transfer. While these are indeed functions of the star products of the external momenta, the phase functions on the annulus are found to be more complex than the simple exponential found in noncommutative scalar field theory [21, 12], and in the vertex operator algebra of open string theory on the disk [8]. The antisymmetric noncommutativity parameter is interpreted as giving a wavefunction renormalization for states in the open string sector. Finally, in section 5, we derive the nonplanar amplitude verifying the presence of a phase proportional to the momentum transfer between boundaries. Using world-sheet duality, we express the string amplitude in terms of closed string variables demonstrating the existence of a pole in the amplitude when the momentum transfer equals the mass of an on-shell closed string state. We summarize our results in the conclusions, making a comparison with the results of other authors in recent papers overlapping our work.

2. Effective string tension and vacuum amplitude

The one-loop vacuum amplitude in the presence of constant B field has been derived

[3] We are careful in distinguishing *finite* renormalization effects, as in the generation of an effective string tension and distinct noncommutativity scale from the bare parameters, α' and $B^{\mu\nu}$, in the world-sheet action, from what we refer to as *wilsonian* renormalization: the rescaling of the infinite bare parameters of a theory to their finite physical values due to quantization of the ultraviolet, or short distance, degrees of freedom. The renormalization of the open string coupling constant — and of the world-sheet cosmological constant, which is renormalized to zero — are wilsonian renormalizations, analogous to wilsonian renormalization in a quantum field theory. Note, also, the clarifications in footnote 11.

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by previous authors using both the path integral method [3] and the background field technique [4]. In this section, we obtain this result following the treatment of the one-loop torus path integral for free strings given in [13], using the same technique to evaluate the Greens function and one-loop scattering amplitudes in following sections of the paper. Thus, we adapt the free closed string calculation in [13] to open and closed string theory in constant background fields in a flat spacetime. We will assume a spacetime metric of euclidean signature, commenting briefly on the necessary modifications of this analysis in the case of a lorentzian metric. We point out that the "open string metric" [8], which appears in the B dependent normalization of the one-loop vacuum amplitude, has a natural world-sheet interpretation as the effective string tension of open and closed string theory in the presence of a background B field. Masses and couplings in the low energy theory are measured in units of the effective tension, $(\alpha'_{\text{eff.}})^{-1/2}$, as opposed to the bare string tension, $(\alpha')^{-1/2}$, with $(\alpha'_{\text{eff.}})^{-1/2} > (\alpha')^{-1/2}$ for non-vanishing B field. It should be emphasized that the effective string tension is a distinct energy scale from the noncommutativity scale, which enters in the vertex operator algebra and scattering amplitudes of the theory. These are discussed in later sections. The effective string scale in a background B field may also be interpreted fruitfully as a finite renormalization of the fundamental string tension.

We begin with the world-sheet action:

$$S = \frac{1}{4\pi\alpha'}\int_M d^2\sigma(\sqrt{g}g^{ab}g_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu})\partial_a X^\mu \partial_b X^\nu + i\int_{\partial M} d\lambda \left(A_\mu \frac{d}{d\lambda}X^\mu\right), \quad (2.1)$$

where $g_{\mu\nu} = \delta_{\mu\nu}$, A_μ is a constant background vector field on the branevolume, and $B_{\mu\nu}$ is a real, antisymmetric, and constant, background tensor field.[4] The B field is restricted to the worldvolume of a Dpbrane with p assumed to be an odd integer. Our results could therefore be extended to the supersymmetric type I string theory with its spectrum of odd p Dpbranes. Note, however, that unlike the type I case, the eigenvalues of the constant B field in the oriented bosonic string theory are not necessarily quantized and can take arbitrary values.

Let us label the directions parallel to the branevolume, $\mu = 0, \ldots, p$. The $25 - p$ directions transverse to the brane volume satisfy Dirichlet boundary conditions. We will assume that the B field has been brought to block diagonal form by a spacetime transformation on the background fields [4], with next-to-diagonal entries $b_{(m)} = B_{2m,2m+1}$, $m = 0, \ldots, (p-1)/2$. Note that the background preserves the Weyl invariance of the action. Thus, gauge fixing to conformal gauge gives the same contribution to the path integral from reparameterization ghost fields as in the quantization of free strings. It can be shown that the $p+1$ dimensional worldvolume of

[4]Note that g need not be a flat spacetime metric, the derivation is identical for arbitrary constant g. In writing eq. (2.1), the constant background fields $g_{\mu\nu}$, $B_{\mu\nu}$, and A_μ, have been assumed dimensionless, while the embedding coordinates and external momenta are dimensionful.

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a Dpbrane is an even dimensional noncommutative space, whose coordinates satisfy the commutation relation [6]:

$$[X^\mu, X^\nu] = 2\pi i\alpha'(\mathcal{M}^{-1}\mathcal{F})^{\mu\nu} \equiv i\Theta^{\mu\nu} . \tag{2.2}$$

The basic observation of a nonvanishing commutator for the zero modes was originally made in [4]. For clarity, we consider a single Dpbrane and set the Maxwell term in the gauge invariant combination of two-form background fields to zero. Thus, $\mathcal{F} \equiv B - 2\pi\alpha' F = B$, and the matrix, $\mathcal{M}$, takes block-diagonal form: $\mathcal{M}_{\mu\nu} = \delta_{\mu\nu} - B_\mu^\lambda B_{\lambda\nu}$. The indices μ, ν, run from $0, \ldots, p$. In section 5 and the conclusions, we return to this issue, explaining briefly the modifications to our analysis for the unoriented bosonic string theory with 2^{13} D25branes and, potentially, a constant background for the Yang-Mills field strength.

The commutation relations given above follow from the Poisson brackets of the X_μ with their canonically conjugate momenta, P_μ, imposing as a constraint the boundary conditions:

$$g_{\mu\nu}\sqrt{g}n^a\partial_a X^\nu + iB_{\mu\nu}t^a\partial_a X^\nu = 0 , \tag{2.3}$$

where n^a and t^a are, respectively, unit normal and tangent vectors to the boundary. Thus, the theory can, at least in principle, be quantized exactly in a canonical framework. In this paper, we will investigate the amplitudes of open and closed string theory in a constant background B field by direct lagrangian path integral evaluation. We will find that the path integral computation enables a derivation of the one-loop scattering amplitudes with the normalization — and all of the background field dependence — explicit. The results obtained are on the same precise footing as the one-loop amplitudes for free strings.

The gauge fixing of the Polyakov path integral is well-known and has been described elsewhere [13]. We begin with the conformal gauge fixed expression for the annulus in constant background B field with both boundaries on the same Dpbrane. It is convenient to use a complex basis, $Z_m = X^{2m} + iX^{2m+1}$, $m = 0, \ldots, (p-1)/2$, and its complex conjugate coordinate, $\bar{Z}_m$, with $b_{(m)} \equiv B_{2m,2m+1}$. Then the commutation relations take the simple form:

$$[Z_m, \bar{Z}_n] = \frac{4\pi\alpha' b_{(m)}}{1 + b_{(m)}^2}\delta_{mn} \equiv 2\theta_{(m)}\delta_{mn} , \tag{2.4}$$

and the boundary conditions on the complex scalars are given by

$$\begin{aligned} l\partial_2 Z_m &= -b_{(m)}\partial_1 Z_m \\ l\partial_2 \bar{Z}_m &= b_{(m)}\partial_1 \bar{Z}_m , \end{aligned} \tag{2.5}$$

The amplitude takes the form:

$$\mathcal{A} = \int_0^\infty \left[\frac{dl}{l}\eta^2\left(\frac{il}{2}\right)\right]\left[\eta\left(\frac{il}{2}\right)\right]^{p-25} \prod_{m=0}^{\frac{p-1}{2}} \int [d\delta Z_m][d\delta \bar{Z}_m] \times$$
$$\times e^{-\left\{\frac{1}{4\pi\alpha'}\int d^2\sigma(\sqrt{g}g^{ab}-\epsilon^{ab}b_{(m)})\partial_a Z_m \partial_b \bar{Z}_m + \mu_0 \int d^2\sigma\sqrt{g}\right\}}, \qquad (2.6)$$

where μ_0 is the bare world-sheet cosmological constant. The measure in the functional integral has been evaluated with respect to the fiducial cylinder metric, $ds^2 = l^2(d\sigma^1)^2 + (d\sigma^2)^2$, where l is the length of the boundary and $0 \leq \sigma^a \leq 1$, $a = 1$, 2. The first factor in square brackets is the remnant of the functional integration over metrics, obtained upon gauge fixing world-sheet reparameterizations [13]. The second factor arises from the functional integration over $25 - p$ Dirichlet embedding coordinates. It remains to compute the contribution from the complex scalars satisfying the boundary conditions given in eq. (2.3).

The laplacian on scalars on the annulus takes the form $\Delta = l^{-2}\partial_1^2 + \partial_2^2$. We expand in a basis of eigenfunctions of the laplacian satisfying the given boundary conditions:

$$\begin{aligned} Z &= z_0 + \sum_{n_1=-\infty}^{\infty}\sum_{n_2=1}^{\infty} z_{n_1 n_2}\Psi_{n_1 n_2} + \sum_{n_1=-\infty}^{\infty} z_{n_1 0}\Psi_{n_1 0} \\ \bar{Z} &= \bar{z}_0 + \sum_{n_1=-\infty}^{\infty}\sum_{n_2=1}^{\infty} \bar{z}_{n_1 n_2}\bar{\Psi}_{n_1 n_2} + \sum_{n_1=-\infty}^{\infty} \bar{z}_{n_1 0}\bar{\Psi}_{n_1 0}\,, \end{aligned} \qquad (2.7)$$

where the complex basis functions, Ψ, take the form:

$$\begin{aligned} \Psi_{n_1 n_2} &= \frac{1}{\sqrt{l}}\exp(2\pi i n_1 \sigma^1)\left(\cos(n_2\pi\sigma^2) - \frac{i\alpha n_1}{n_2}\sin(n_2\pi\sigma^2)\right) \\ \bar{\Psi}_{n_1 n_2} &= \frac{1}{\sqrt{l}}\exp(2\pi i n_1 \sigma^1)\left(\cos(n_2\pi\sigma^2) + \frac{i\alpha n_1}{n_2}\sin(n_2\pi\sigma^2)\right) \\ \Psi_{n_1 0} &= \frac{1}{\sqrt{l}}\exp\left[2\pi i n_1\left(\sigma^1 - \frac{b}{l}\sigma^2\right)\right] \\ \bar{\Psi}_{n_1 0} &= \frac{1}{\sqrt{l}}\exp\left[2\pi i n_1\left(\sigma^1 + \frac{b}{l}\sigma^2\right)\right], \end{aligned} \qquad (2.8)$$

and the parameter $\alpha = 2b/l$. The eigenfunctions correspond, respectively, to eigenvalues:

$$\begin{aligned} \omega_{n_1 n_2} &= \frac{4\pi^2}{l^2}n_1^2 + \pi^2 n_2^2 \\ \omega_{n_1 0} &= \frac{4\pi^2 n_1^2}{l^2}(1+b^2)\,, \end{aligned} \qquad (2.9)$$

where the subscripts take values in the range $-\infty \leq n_1 \leq \infty$, $n_2 \geq 1$. In the limit of zero B field, we recover the eigenfunctions and eigenspectrum of a complex scalar satisfying Neumann boundary conditions. A natural choice of reparameterization invariant norm on the space of complex eigenfunctions is given by the orthogonality relations:

$$\begin{aligned}\int d^2\sigma\sqrt{g}\bar{\Psi}_{n_1'n_2'}\Psi_{n_1n_2} &= C_{n_1n_2}\delta_{n_1',-n_1}\delta_{n_2'n_2}\\ \int d^2\sigma\sqrt{g}\bar{\Psi}_{n_1'0}\Psi_{n_10} &= C_{n_10}\delta_{n_1',-n_1}\\ \int d^2\sigma\sqrt{g}\bar{\Psi}_{n_1'n_2}\Psi_{n_10} &= \int d^2\sigma\sqrt{g}\bar{\Psi}_{n_1'0}\Psi_{n_1n_2} = 0\,,\end{aligned} \tag{2.10}$$

where the orthogonality coefficients are given by:

$$C_{n_1n_2} = \frac{1}{2}a\left(1 - \frac{n_1^2\alpha^2}{n_1^2\alpha^2}\right), \qquad C_{n_10} = e^{-i\pi n_1\alpha}\frac{\sin(\pi n_1\alpha)}{\pi n_1\alpha}\,. \tag{2.11}$$

Note that the coefficients are not positive definite. This is the first indication of the unusual properties of open string theory in a constant background B field in a euclidean spacetime: we will find that while the orthogonality coefficients drop out of the expression for the vacuum amplitude, they play a crucial role in the scattering amplitude. It should be noted that in a spacetime of lorentzian signature — with true *spacetime* noncommutativity, the orthogonality coefficients would be positive definite. We return to this point in the conclusions.

The normalization of the vacuum amplitude will be obtained as in [13]. We use the reparameterization invariant norm for a free complex scalar:

$$\begin{aligned}|\delta Z|^2 &= \int d^2\sigma\sqrt{g}(\delta\bar{Z})(\delta Z)\\ &= \sum_{n_1=-\infty}^{\infty}\sum_{n_2=1}^{\infty} C_{n_1n_2}\delta\bar{z}_{-n_1n_2}\delta z_{n_1n_2} + \sum_{n_1=-\infty}^{\infty} C_{n_10}\delta\bar{z}_{-n_10}\delta z_{n_10}\,,\end{aligned} \tag{2.12}$$

substituting the basis functions given in eq. (2.8). Following [13], we assume that the measure in the tangent space to the space of embeddings is ultralocal. Namely, the functional integral over embeddings, $Z(\sigma^a)$, is the product of ordinary integrals defined at some base point, σ^a, on the world-sheet, followed by an integration of the location of the base point in the domain $0 \leq \sigma^a \leq 1$. The measure for the zero modes is obtained in the usual way setting:

$$\begin{aligned}&\int d\delta Z d\delta\bar{Z}\; e^{-|\delta Z|^2/2} \equiv 1\\ &= \int [dz_0][d\bar{z}_0]\prod_{n_1,n_2}\int dz_{n_1n_2}d\bar{z}_{n_1n_2}dz_{n_10}d\bar{z}_{n_10}\; J\; e^{-|\delta Z|^2/2}\,,\end{aligned} \tag{2.13}$$

where J is the jacobian from the change of variables in eq. (2.12). Performing the gaussian integrations explicitly determines this as:

$$J = \frac{l}{2\pi}\prod_{n_2=0}^{\infty}\prod_{n_1=-\infty}^{\infty\prime}\left(\frac{C_{n_1n_2}}{2\pi}\right). \tag{2.14}$$

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Thus, the path integral for a single complex scalar in constant background field takes the form:

$$\int [d\delta Z][d\delta \bar{Z}] e^{-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\ \bar{Z}\Delta Z} = \left(\frac{l}{2\pi}\right) \prod_{n_1=-\infty}^{\infty\prime} \prod_{n_2=0}^{\infty} \frac{C_{n_1 n_2}}{2\pi} \times$$
$$\times \int dz_{n_1 n_2} d\bar{z}_{n_1 n_2} e^{-\left\{\frac{1}{4\pi\alpha'}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=0}^{\infty}\omega_{n_1 n_2} C_{n_1 n_2}\bar{z}_{n_1 n_2} z_{n_1 n_2}\right\}}, \quad (2.15)$$

where the prime denotes exclusion of the $n_1 = n_2 = 0$ mode from the infinite product. Performing the integrals as above, we obtain the following expression for the amplitude:

$$\mathcal{A} = iV_{p+1} \int_0^\infty \left[\frac{dl}{l}\eta^2\left(\frac{il}{2}\right)\right]\eta\left(\frac{il}{2}\right)^{-(25-p)}\left(\frac{l}{2\pi}\right)^{\frac{p+1}{2}} \times$$
$$\times \left[\prod_{m=0}^{(p-1)/2}\prod_{n_1=-\infty}^{\infty}\prod_{n_2=0}^{\infty}(2\pi)\frac{\omega_{n_1 n_2}^{(m)}}{4\pi^2\alpha'}\right]^{-1}. \quad (2.16)$$

Note that the jacobian from the change of variables is cancelled against a similar term arising from the gaussian integration, such that the orthogonality coefficients are absent from the final expression. The functional determinant of the laplacian is evaluated using zeta function regularization. The result is:

$$\mathcal{A} = iV_{p+1}\det\left(\mathbf{1}+\mathbf{B}\right)\int_0^\infty \frac{dl}{l}\left(4\pi^2\alpha' l\right)^{-\frac{p+1}{2}}\eta\left(\frac{il}{2}\right)^{-24}, \quad (2.17)$$

in agreement with the references [3, 4]. The renormalized value of the world-sheet cosmological constant has been set to zero as in [13]. The sum over the $n_2 > 0$ modes gives $p+1$ powers of the η function, analogous to the result for $p+1$ free scalars with Dirichlet boundary conditions. The B dependent pre-factor in the amplitude arises from the $n_2 = 0$ modes as follows:

$$\prod_{m=0}^{(p-1)/2}\prod_{n_1=-\infty}^{\infty\prime}\left[\frac{\omega_{n_1 0}^{(m)}}{2\pi\alpha'}\right]^{-1} = \prod_{m=0}^{(p-1)/2}\prod_{n_1=1}^{\infty}\left[\frac{\omega_{n_1 0}^{(m)}}{2\pi\alpha'}\right]^{-2}$$
$$= \left(2\pi\alpha' l^2\right)^{-\frac{p+1}{2}}\det(\mathbf{1}+\mathbf{B}), \quad (2.18)$$

where the prime denotes exclusion of the $n_1 = 0$ mode from the infinite product, and we have used the identity:

$$\ln\det\{\cdots\} = -\lim_{s\to 0}\frac{d}{ds}\left\{\left[\frac{2\pi}{l^2\alpha'}(1+b^2)\right]^{-s}\sum_{n_1=1}^{\infty} n_1^{-2s}\right\}$$
$$= \ln\zeta(0)\ \ln\ \left[\frac{2\pi}{l^2\alpha'}(1+b^2)\right] - 2\zeta'(0)$$
$$= \frac{1}{2}\ln\left[\frac{2\pi l^2\alpha'}{(1+b^2)}\right]. \quad (2.19)$$

Here, $\zeta(0)$ is the continuation of the ordinary Riemann zeta function to its value with zero argument. In the limit of zero B field, we recover the contribution from the $n_2 = 0$ modes of $p+1$ real scalars satisfying Neumann boundary conditions [13].

It is interesting to interpret the B dependent normalization of the one-loop vacuum amplitude given in eq. (2.17). Notice that the only change from the one-loop vacuum amplitude for free strings is that the fundamental string tension, $\tau_F = \frac{1}{2\pi\alpha'}$, has been rescaled to an *effective* string tension:

$$\begin{aligned}\tau_{\text{eff.}}^{(p+1)/2} &= (2\pi\alpha'_{\text{eff.}})^{-\frac{p+1}{2}} \\ &\equiv \left[\det(\mathbf{1}+\mathbf{B})(2\pi\alpha')^{-\frac{p+1}{2}}\right]. \end{aligned} \tag{2.20}$$

Thus, open string theory in a constant background B field probes shorter distance scales or, equivalently, higher momentum scales, with an ultraviolet cutoff of $O(\sqrt{\alpha'_{\text{eff.}}})$. The coupling to constant background fields effectively *raises* the string scale. We will see this feature repeated in the scattering amplitudes: the fundamental string scale does not appear in the scattering amplitudes except in the particular combination given in eq. (2.20). In addition, new parameters associated with noncommutativity are present in the planar and nonplanar scattering amplitudes. The same is true of the low energy spacetime actions for massless fields derived from the string theory. They are characterized by the appearance of two finitely distinct energy scales: $(\alpha'_{\text{eff.}})^{-1/2}$, and a noncommutativity scale, $\Theta^{-1/2}$, to be defined below. We will find that while $(\alpha'_{\text{eff.}})^{-1/2}$ is always above the fundamental string scale, the noncommutativity scale can be lower or higher depending on the specific values of the background fields. We return to this point in the conclusions.

3. Greens function on the annulus

The main ingredient required for the computation of planar and nonplanar one-loop amplitudes is the bulk Greens function for two sources on the annulus. In this section, we perform this computation, examining also the divergences in the Greens function for closely separated sources on the boundary of the world-sheet. We will find that the requisite renormalization of the open string coupling proceeds much the same way as for free string theory, except for the appearance of noncommutativity parameters in the *finite* part of the Greens function. This will lead to momentum dependent phase functions normalizing the planar and nonplanar scattering amplitudes. These are derived in the following section.

Consider the annulus with N open string tachyon vertex operator insertions:

$$V(k_i) = g\int_{\partial M} ds\sqrt{g}e^{i(\mathbf{k}\cdot\bar{\mathbf{Z}}+\bar{\mathbf{k}}\cdot\mathbf{Z})}\,, \tag{3.1}$$

corresponding to delta function sources, $J^m(\sigma^a) = k_i^m\delta(\sigma^1-\sigma_i^1)$, $\bar{J}^m(\sigma^a) = \bar{k}_i^m\delta(\sigma^1-\sigma_i^1)$, inserted at locations σ_i^1, $i=1,\ldots,N$, on the boundary of the world-sheet, with momentum, $\sum_{m=0}^{(p-1)/2} k_i^m \bar{k}_i^m = \mathbf{p}_i^2/4$. The factor of 4 comes from our use of a complex basis for momentum vectors defined below. Here g is the open string coupling. The contribution from a single complex scalar to the path integral over world-sheets coupled to external sources takes the form [9]:

$$
\begin{aligned}
&\int [d\delta Z][d\delta \bar{Z}] e^{\left\{\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\ \bar{Z}\Delta Z + i\int d^2\sigma\sqrt{g}(\bar{J}Z+J\bar{Z})\right\}} = \\
&= \frac{l}{2\pi}\prod_{n_1=-\infty}^{\infty}\prod_{n_2=0}^{\infty}\left(\frac{C_{n_1n_2}}{2\pi}\right)\times\int dy_{n_1n_2}d\bar{y}_{n_1n_2}\times \\
&\quad\times e^{-\left\{\frac{1}{4\pi\alpha'}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=0}^{\infty}\omega_{n_1n_2}C_{n_1n_2}\bar{y}_{-n_1n_2}y_{n_1n_2}\right\}}\times \\
&\quad\times e^{-\left\{4\pi\alpha'\sum_{n_1=-\infty}^{\infty}\sum_{n_2=0}^{\infty}\left[(\omega_{n_1n_2}C_{n_1n_2})^{-1}\int_M d^2\sigma d^2\sigma'\bar{J}(\sigma)J(\sigma')\bar{\Psi}_{-n_1n_2}(\sigma')\Psi_{n_1n_2}(\sigma)\right]\right\}},
\end{aligned}
\tag{3.2}
$$

where the y oscillators are defined by the shift:

$$
\begin{aligned}
y_{n_1n_2} &= z_{n_1n_2} - \frac{i4\pi\alpha'}{\omega_{n_1n_2}C_{n_1n_2}}\int d^2\sigma\sqrt{g}J\bar{\Psi}_{-n_1n_2}(\sigma) \\
\bar{y}_{-n_1n_2} &= \bar{z}_{-n_1n_2} - \frac{i4\pi\alpha'}{\omega_{n_1n_2}C_{n_1n_2}}\int d^2\sigma\sqrt{g}\bar{J}\Psi_{n_1n_2}(\sigma)\,.
\end{aligned}
\tag{3.3}
$$

Substituting in the amplitude and performing the gaussian integrations as before gives the result:

$$
\mathcal{A}(1,\ldots,N) = \int_0^\infty \frac{dl}{l}N_B\left[\eta\left(\frac{il}{2}\right)\right]^{-24} e^{-\sum_{i,j=1}^{N}\sum_{m,n=0}^{(p-1)/2}k_i^m\bar{k}_j^n G'_{mn}(\sigma_i^a,\sigma_j^a)-\mu_1\int d^2\sigma\sqrt{g}}\,.
\tag{3.4}
$$

We include a term in the exponent proportional to the world-sheet cosmological constant in order to soak up any divergent contributions to the Green's function. These will be understood as being absorbed in a renormalization of the bare string coupling constant. We use a complex momentum basis with vectors, $2k_m = p_{2m}+ip_{2m+1}$, $2\bar{k}_m = p_{2m} - ip_{2m+1}$, $m = 0,\ldots,(p-1)/2$, and where the p_μ are real momentum vectors. With this convention, the symmetric and antisymmetric bilinear products can be written as

$$
\begin{aligned}
\frac{1}{2}\left(p_{2m}^i p_{2m}^j + p_{2m+1}^i p_{2m+1}^j\right) &= k_i^m\bar{k}_j^m + \bar{k}_i^m k_j^m \\
\frac{i}{2}\left(p_{2m+1}^i p_{2m}^j - p_{2m+1}^j p_{2m}^i\right) &= k_i^m\bar{k}_j^m - \bar{k}_i^m k_j^m\,,
\end{aligned}
\tag{3.5}
$$

for each value of m. The Greens function for a complex scalar in the presence of external sources takes the form

$$
G'(\sigma_i^a,\sigma_j^a) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=0}^{\infty}\frac{4\pi\alpha'}{\omega_{n_1n_2}C_{n_1n_2}}\bar{\Psi}_{-n_1n_2}(\sigma_i^a)\Psi_{n_1n_2}(\sigma_j^a)\,.
\tag{3.6}
$$

It satisfies Laplace's equation in the cylinder metric with the addition of a background charge given by the normalization of the zero mode derived above [9]. Note that the integration over zero modes, z_0, $\bar{z}_0$, gives the usual delta function in momentum space in the large volume limit, and we have defined the normalization factor:

$$N_B = i(2\pi)^{p+1} g^N \delta\left(\sum_{i=1}^{N} \mathbf{p}_i\right) (4\pi^2\alpha' l)^{-(p+1)/2} \det(\mathbf{1}+\mathbf{B}) \prod_{i=1}^{N} \int ds_i \sqrt{g} \,. \tag{3.7}$$

The derivation of the Greens function on the annulus is tedious, although straightforward, and is sketched in the appendix. For a single complex scalar and non-coincident sources, $i \neq j$, we obtain the result:

$$\begin{aligned} G'(\sigma_i^a, \sigma_j^a) = & -\frac{2\alpha'}{1+b^2} \ln \frac{|\Theta_1(\nu_+, \tau)|^2}{|\eta(\tau)|^2} - \\ & -\frac{2i\alpha' b}{1+b^2} \ln \frac{\Theta_1(\nu_+, \tau)}{\Theta_1^\star(\nu_+, \tau)} - \alpha' \ln |\frac{\Theta_1(\nu_-, \tau)}{\Theta_1(\nu_+, \tau)}|^2 + \\ & + \frac{2\pi\alpha' l}{1+b^2}\left[\sigma^2 - \frac{2b\sigma}{l} + \frac{2b^2}{3l^2}\right]. \end{aligned} \tag{3.8}$$

The notation is as follows:

$$\begin{aligned} \nu_\pm &= \frac{1}{2}(il\sigma_-^1 + \sigma_\pm^2)\,, \quad \tau = \frac{il}{2}\,, \\ \sigma_\pm^a &= \sigma_i^a \pm \sigma_j^a\,, \quad \sigma = \sigma_-^1 + \frac{b}{l}\sigma_+^2\,. \end{aligned} \tag{3.9}$$

The Greens function for $(p+1)/2$ scalars coupled to distinct constant background fields, $b_{(m)}$, $m = 0, \ldots, (p-1)/2$, is the direct sum of expressions of the form given in eq. (3.8).[5]

The Greens function for two sources on the boundary of an annulus can be straightforwardly extracted from this expression. We will obtain the limiting behavior of the Greens function for sources with small separation as follows. Setting $\sigma_i^2 = \sigma_j^2 = 0,\ 1$, gives the Greens functions with two sources on the same boundary.

$$\begin{aligned} G'_P(\sigma_i, \sigma_j) = & -\frac{2\alpha'}{1+b^2} \ln \frac{|\Theta_1(\frac{il\sigma_{ij}}{2}, \frac{il}{2})|^2}{|\eta(\frac{il}{2})|^2} + \\ & + \frac{2\pi\alpha' l}{1+b^2}\left[(\sigma_{ij})^2 - \frac{2b\sigma_{ij}}{l} + \frac{2b^2}{3l^2}\right] + \\ & + \frac{2\pi\alpha' b}{1+b^2}\frac{\sigma_{ij}}{|\sigma_{ij}|} \quad \sigma^2 = 0 \end{aligned}$$

[5] An expression for the bulk Greens function has been derived using the background field method in [17]. Comparison with [17, eq. (3.8)] shows agreement for the first three terms in eq. (3.8). The derivation in [17] is missing the contributions from the constant modes, including a piece that cancels one of the terms present in ref. [17, eq. (3.8)]. Notice that the term linear in σ_{ij}, which will contribute an external momentum dependent phase in the planar amplitude, and which has opposite sign for the $\sigma^2 = 0,\ 1$, boundaries, is absent in [15]–[18].

$$G'_P(\sigma_i, \sigma_j) = -\frac{2\alpha'}{1+b^2} \ln \frac{|\Theta_1(\frac{il\sigma_{ij}}{2}, \frac{il}{2})|^2}{|\eta(\frac{il}{2})|^2} + $$
$$+\frac{2\pi\alpha' l}{1+b^2}\left[(\sigma_{ij})^2 + \frac{2b\sigma_{ij}}{l} + \frac{2b^2}{3l^2}\right] -$$
$$-\frac{2\pi\alpha' b}{1+b^2}\frac{\sigma_{ij}}{|\sigma_{ij}|} \quad \sigma^2 = 1\,. \tag{3.10}$$

We are henceforth suppressing the superscript on $\sigma^1_- \equiv \sigma_{ij}$. We comment that the phases in these expressions have been determined directly from the summations in the appendix, rather than from the compact expression for the bulk propagator given in eq. (3.8). Consequently, there is no ambiguity in the choice of branch for the logarithm function in the second term in eq. (3.8).

We now consider self-contractions. Since the Greens function is diagonal in the complex basis, it suffices to consider a single complex scalar. The exponent in the expression for the N-point scattering amplitude given in eq. (3.4) can be rewritten as follows [9]:

$$\sum_{i,j=1}^N k_i\bar{k}_j G'_P(\sigma_i,\sigma_j) = \sum_{i\neq j;\ i,j=1}^N k_i\bar{k}_j G'_P(\sigma_i,\sigma_j) + \sum_{i=1}^N k_i\bar{k}_i G'_P(\sigma_i,\sigma_i)$$
$$= \frac{1}{2}\sum_{i\neq j;\ i,j=1}^N \Big[(k_i\bar{k}_j + k_j\bar{k}_i) G'^{(e)}_P(\sigma_i,\sigma_j) +$$
$$+ (k_i\bar{k}_j - k_j\bar{k}_i) G'^{(o)}_P(\sigma_i,\sigma_j)\Big] - \tag{3.11}$$
$$-\frac{1}{2}\sum_{i\neq j;i,j=1}^N (k_i\bar{k}_j + k_j\bar{k}_i) G'_P(\sigma_i,\sigma_i)\,, \tag{3.12}$$

where we have used momentum conservation in obtaining the last step. Note that we could further simplify this expression by writing the double sums as summations over i, j, with restriction $i > j$, or vice versa. The form given here will be found convenient in establishing, in section 4.1, the interpretation of the momentum dependent phase functions normalizing planar, and nonplanar, amplitudes as the wavefunction renormalization of open string vertex operators. Note that we have distinguished terms in the boundary Greens function that are even (odd) under the interchange of i and j by the superscripts (e), (o). The summation incorporates both the symmetric and the antisymmetric momentum bilinears defined in eq. (3.5). The presence in the Greens function of both the symmetric and antisymmetric invariants in the noncommutativity parameters, will carry through to the momentum dependent phases normalizing the amplitude.

We focus on the self-contractions in the last term, obtained as a limit of the Greens function for closely separated sources. In the zero B field limit, this takes the general form:

$$\lim_{\sigma_i\to\sigma_j} G'_P(\sigma_i,\sigma_j) = -2\alpha' \ \ln \ d^2(|\sigma_{ij}|) + f(\sigma_i,\sigma_j)\,, \tag{3.13}$$

where d is the distance between the sources as measured on the world-sheet, and the function f is finite in the limit $\sigma_i = \sigma_j$. Let us evaluate the leading short distance singularity in the Greens function in the presence of a constant background B field. In the limit of small separations, eq. (3.10) gives

$$\begin{aligned} G'_P(\sigma_i,\sigma_i) = & -\frac{2\alpha'}{1+b^2}\ln|l\sigma_{ij}|^2 - \\ & -\frac{2\alpha'}{1+b^2}\ln\left|\pi\left(\eta\left(\frac{il}{2}\right)\right)^2\right|^2 + \frac{4\pi\alpha' b^2}{3l(1+b^2)}\,, \qquad \sigma_i^2 = 0,1\,. \end{aligned} \tag{3.14}$$

The leading short distance divergence in eq. (3.14) has the same form as on the boundary of the disk [9, 8], except that d is specified with respect to the fiducial metric on the annulus. Note that one can define a regulated Greens function by subtracting the divergence:

$$G_P^{\text{reg.}}(\sigma_i,\sigma_i) = G'_P(\sigma_i,\sigma_i) + \frac{2\alpha'}{1+b^2}\lim_{\sigma_i\to\sigma_j}\ln|l\sigma_{ij}|^2\,, \tag{3.15}$$

on either boundary, making the same subtraction in eqs. (3.12) and (3.6). The regulated Greens functions enter into the derivation of the scattering amplitudes in the next section. The divergent terms are to be understood as having been absorbed in a renormalization of the bare open string coupling [13].

Finally, we can write down the Greens function with sources on different boundaries:

$$\begin{aligned} G'_{NP}(\sigma_i,\sigma_j) = & -\frac{2\alpha'}{1+b^2}\ln\left|\frac{\Theta_2(\frac{il\sigma_{ij}}{2},\frac{il}{2})}{\eta(\frac{il}{2})}\right|^2 + \\ & +\frac{2\pi\alpha' l}{1+b^2}\left[(\sigma_{ij})^2 - \frac{b^2}{3l^2}\right] \qquad \sigma_+^2 = 1\,. \end{aligned} \tag{3.16}$$

Notice that the Greens function with sources on different boundaries is even under their interchange and, as expected, there are no short distance divergences.

4. Wavefunction renormalization and planar amplitudes

Consider now the planar amplitude with N tachyon vertex operator insertions on a single boundary of the annulus. The amplitude can be put in a more transparent form as follows. Following [8, 7, 12], it is natural to identify symmetric, and antisymmetric, momentum bilinears for any given pair of vertex operators, (V_i, V_j), with *complex* momenta, $(\mathbf{k}_i, \mathbf{k}_j)$:

$$
\begin{aligned}
\mathbf{k}_i \cdot \bar{\mathbf{k}}_j &= \sum_{m=0}^{(p-1)/2} \frac{\alpha'}{1+b_{(m)}^2}(k_i^m \bar{k}_j^m + k_j^m \bar{k}_i^m) \\
\mathbf{k}_i \circ \bar{\mathbf{k}}_j &= \sum_{m=0}^{(p-1)/2} \frac{\pi\alpha' b_{(m)}^2}{1+b_{(m)}^2}(k_i^m \bar{k}_j^m + k_j^m \bar{k}_i^m) \\
\mathbf{k}_i \times \bar{\mathbf{k}}_j &= \sum_{m=0}^{(p-1)/2} \frac{2\pi\alpha' b_{(m)}}{1+b_{(m)}^2}(k_i^m \bar{k}_j^m - k_j^m \bar{k}_i^m) \\
&= \sum_{m=0}^{(p-1)/2} \theta_{(m)}(k_i^m \bar{k}_j^m - k_j^m \bar{k}_i^m)\,. \qquad (4.1)
\end{aligned}
$$

Note that one can construct two distinct symmetric bilinear products in the momentum by taking appropriate powers of B. Only the first appears in the planar amplitude. The antisymmetric bilinear product accompanies terms in the Greens function that are odd under the interchange of i with j. Both symmetric and antisymmetric forms appear in the scattering amplitudes, and are also present in the vertex operator algebra of string theory in a background B field. The spacetime invariants implicit in the bilinear products given in eq. (4.1) are manifest in the alternative basis: [6]

$$
\begin{aligned}
\mathbf{k}_i \cdot \bar{\mathbf{k}}_j &= \frac{\alpha'}{2}\sum_{\mu,\nu=0}^{p} (\mathcal{M}^{-1})^{\mu\nu} p_\mu^i p_\nu^j \\
\mathbf{k}_i \circ \bar{\mathbf{k}}_j &= -\frac{1}{8\pi\alpha'}\sum_{\mu,\nu=0}^{p} \Theta^{\mu\sigma}(\mathcal{M})_{\sigma\lambda}\Theta^{\lambda\nu}\; p_\mu^i p_\nu^j \\
\mathbf{k}_i \times \bar{\mathbf{k}}_j &= -\frac{i}{2}\sum_{\mu,\nu=0}^{p} \Theta^{\mu\nu}\; p_\mu^i p_\nu^j\,, \qquad (4.2)
\end{aligned}
$$

where spacetime indices are raised and lowered using the metric, $g_{\mu\nu} = \delta_{\mu\nu}$, and $\mathcal{M}^{-1}$, $\Theta_{\mu\nu}$ are defined by the spacetime commutation relations given in eq. (2.2). Note that the first of the spacetime invariants can be identified with the "open string metric" of [8]. We have used eq. (3.5) to relate the real and complex momentum bases. The field theory limit corresponds to taking α' to zero, keeping fixed the dimensionless variables, B, $\mathcal{M}$, and Θ/α'. [7]

[6] Note that $\mathbf{k} \circ \bar{\mathbf{k}}$ is identical in form to the spacetime invariant, $\mathbf{p} \circ \mathbf{p}$, appearing in the noncommutative field theory analysis of [12]. However, it is *dimensionless*.

[7] This does *not* coincide with the scaling limit described in [8] since, with our conventions, B is dimensionless.

It is convenient to perform all calculations in the complex basis. Substituting for the Greens function in eq. (3.6), the one-loop planar amplitude takes the form:

$$\mathcal{A}_P(1,\ldots,N) = i\tilde{g}^N \delta\left(\sum_{i=1}^{N}\mathbf{p}_i\right)(\alpha')^{-\frac{p+1}{2}}\det(\mathbf{1}+\mathbf{B})\left[\prod_{r=1}^{N}\int d\sigma_r\right]\times$$
$$\times\int_0^\infty \frac{dl}{l} e^{-\mu_2\int d^2\sigma\sqrt{g}} \times \frac{l^{N-(p+1)/2}}{\eta(\frac{il}{2})^{24}}\times$$
$$\times\prod_{i\neq j;\ i,j=1}^{N}\left[e^{\mathbf{k}_i\times\bar{\mathbf{k}}_j\left(\sigma_{ij}-\frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)} f_1(l;\sigma_{ij})\right], \tag{4.3}$$

where we have defined the functions:

$$f_a(l;\sigma) \equiv e^{-\pi l\sigma_{ij}^2\mathbf{k}_i\cdot\bar{\mathbf{k}}_j}\left|\frac{\Theta_a(\frac{il}{2}\sigma_{ij},\frac{il}{2})}{\pi\eta^3(\frac{il}{2})}\right|^{2\mathbf{k}_i\cdot\bar{\mathbf{k}}_j}, \tag{4.4}$$

and included a term in the exponent proportional to the world-sheet cosmological constant in order to soak up any divergent terms generated in the renormalized amplitude. We have used the form of the boundary propagator given in the first of eqs. (3.10). Notice that the difference in the sign of the odd part of the Greens function is correlated with the ordering of the sources on the boundary. Note also that the bare open string coupling, g, has been replaced by the multiplicatively renormalized string coupling, $\tilde{g}$, [13] as discussed in the previous section. The prefactor linear in σ_{ij} can be recognized using eq. (4.2):

$$\exp\left[(\mathbf{k}_i\times\bar{\mathbf{k}}_j)\sigma_{ij}\right] = \exp\left[-\frac{i}{2}\Theta^{\mu\nu}p_\mu^i p_\nu^j\sigma_{ij}\right]. \tag{4.5}$$

The star product, denoted by the $\times$ symbol in eqs. (4.2), makes a natural appearance in the antisymmetric bilinear product of the external momenta.

Consider now the field theory limit in which only the massless open string mode propagates around the loop. This is extracted from the expression above by isolating the $O(q^0)$ term in an expansion of the integrand in powers of $q = e^{-\pi l}$:

$$\lim_{l\to\infty}\left\{\eta^{-24}\left|\frac{\Theta_1(\frac{il}{2}\sigma_{ij},\frac{il}{2})}{\pi\eta^3(\frac{il}{2})}\right|^{2\mathbf{k}_i\cdot\bar{\mathbf{k}}_j}\right\} =$$
$$= e^{\pi l}\left(1+24e^{-\pi l}\right)\left|\frac{2}{\pi}\sinh\left(\frac{1}{2}\pi l\sigma_{ij}\right)\right|^{2\mathbf{k}_i\cdot\bar{\mathbf{k}}_j}\times$$
$$\times\left(1-\left[8\mathbf{k}_i\cdot\bar{\mathbf{k}}_j\sinh^2\left(\frac{1}{2}\pi l\sigma_{ij}\right)\right]e^{-\pi l}\right) \tag{4.6}$$

upto $O(e^{-\pi l})$ corrections. We suppress the leading term of $O(e^{\pi l})$, which corresponds to the open string tachyon propagating in the loop, focusing on the $O(1)$ correction.

In this limit, the scattering amplitude takes the form:

$$
\begin{aligned}
&\mathcal{A}_P|_{\text{massless}} = \\
&\quad = i\tilde{g}^N \delta\left(\sum_{i=1}^N \mathbf{p}_i)(\alpha'\right)^{-(p+1)/2} \det(\mathbf{1}+\mathbf{B}) \left[\prod_{r=1}^N \int d\sigma_r\right] \times \\
&\qquad \times \int_0^\infty dl l^{N-(p+3)/2} e^{-\mu_2 \int d^2\sigma\sqrt{g}} \prod_{i\neq j;\ i,j=1}^N e^{(\mathbf{k}_i\times\bar{\mathbf{k}}_j)\left(\sigma_{ij}-\frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)} \times \\
&\qquad \times \left\{ e^{-\pi l \sigma_{ij}^2 \mathbf{k}_i\cdot\bar{\mathbf{k}}_j} \left|\frac{2}{\pi}\sinh\left(\frac{1}{2}\pi l\sigma_{ij}\right)\right|^{2\mathbf{k}_i\cdot\bar{\mathbf{k}}_j} \left[24 - 8\mathbf{k}_i\cdot\bar{\mathbf{k}}_j \sinh^2\left(\frac{1}{2}\pi l\sigma_{ij}\right)\right]\right\}. \quad (4.7)
\end{aligned}
$$

It should be emphasized that the corrections from massive open string states circulating in the loop do not change the basic form of this result, the single term given here being replaced by a series of the form:

$$\sum_{n=0}^{\infty} F_n^{(\text{open})}\left(\sinh\left(\frac{1}{2}\pi l\sigma_{ij}\right)\right) e^{-(\sigma_{ij}^2+n)\pi l}\,, \tag{4.8}$$

where $F_n^{(\text{open})}$ is a polynomial function of the $\text{Sinh}(\frac{1}{2}\pi l\sigma_{ij})$, i,$j = 1,\ldots,N$. The nth term in the series represents the contribution from states in the nth mass level of the open string spectrum. The effective string tension introduced earlier defines a natural scale with respect to which we can measure external momentum. We will now restrict to on-shell external momenta setting

$$\mathbf{k}_i\cdot\bar{\mathbf{k}}_i = \sum_{m=0}^{(p-1)/2} \frac{2\alpha'}{1+b_{(m)}^2} k_i^m \bar{k}_i^m = \frac{n_i}{2}\,, \tag{4.9}$$

where the n_i take integer values. Vertex operators with $n_i = 1$ correspond to on-shell open string tachyons, with the delta function imposing momentum conservation. In contrast to the field theory limit of the planar amplitudes of free string theory in zero background B field and ordinary commutative space, the noncommutative amplitude is characterized by the presence of exponentiated phases:

$$e^{(\mathbf{k}_i\times\bar{\mathbf{k}}_j)\left(\sigma_{ij}-\frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)} = e^{-\frac{1}{2}\sum_{\mu,\nu=0}^{p} i\Theta^{\mu\nu} p_\mu^i p_\nu^j \left(\sigma_{ij}-\frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)}, \tag{4.10}$$

for all i, $j = 1,\ldots,N$. The antisymmetric star products of the external momenta make a natural appearance, in accordance with the expectations of noncommutative scalar field theory [21]. In order to make a more direct comparison with the results of noncommutative field theory, it is helpful to explore further the external momentum dependent normalization of the planar amplitude.

A natural interpretation of a momentum dependent phase function normalizing the planar amplitude is that it represents a wavefunction renormalization. To ascertain the form of such a phase function it is helpful to carry out the integrations over the modular parameter, l, and the σ_i explicitly, making a suitably simple choice of kinematics.[8] We will show now that upon imposing the on-shell conditions on the external momenta, and in the restricted kinematics of forward scattering, the integrals over the modular parameter, l, and the σ_i can in fact be performed explicitly. The result clarifies that the momentum dependent phase function normalizing the amplitude obtained as a result of both integrations, can indeed be interpreted as a wavefunction renormalization of the open string vertex operators.

4.1 Forward scattering and the decoupling limit

Consider the limit of forward scattering with $\cos(\phi_{ij}) = \pm 1$, for every $i, j = 1, \ldots, N$, in addition to the on-shell conditions imposed in eq. (4.9). To accommodate momentum conservation with zero momentum transfer, $\sum_{i=1}^{N} \mathbf{p}_i = 0$, and finite, even, N, we could, for example, make the symmetric choice of $N/2$ parallel incoming and $N/2$ parallel outgoing on-shell tachyons. Alternatively, we could consider $N-1$ parallel incoming on-shell tachyons, and a single outgoing off-shell tachyon with $2\mathbf{k}_N \cdot \bar{\mathbf{k}}_N = N - 1$. For either choice, a glance at eq. (4.7) shows that the integration over l can be performed explicitly. The σ_i integrals can then be performed by the iterative method described below. However, it must be kept in mind that zero momentum transfer implies the regime, $\mathbf{p}^2 \ll \alpha'_{\text{eff.}}$, which is properly examined in the closed string channel, as in the next section. We will instead examine here the forward scattering of N on-shell tachyons, where N is a large number. Momentum conservation implies that the forward momentum is carried away by the closed string state which is, consequently, infinitely massive for sufficiently large N. In the limit of large momentum transfer, the decoupling of the massive closed string states from the massless states in the field theory limit of the open string sector is natural, and the restriction to the $O(1)$ term in the expansion in $e^{-\pi l}$ increasingly well-justified. In the limit of infinite momentum transfer, the approximation becomes exact. We will therefore consider the forward scattering of N parallel on-shell tachyons as a simple and concrete realization of the planar amplitude in the decoupling limit. In section 5, we will consider the opposite regime of zero momentum transfer.

Setting $2\mathbf{k}_i \cdot \bar{\mathbf{k}}_j = 1$ for all $i, j = 1, \ldots, N$, the planar amplitude takes the simple form:

$$\mathcal{A}_P|_{\text{decoup.}} = i\tilde{g}^N \delta\left(\sum_{i=1}^{N} \mathbf{p}_i - \mathbf{p}\right) \prod_{i,j=1}^{N} \delta(\mathbf{p}_i \cdot \mathbf{p}_j - 1)(\alpha')^{-\frac{p+1}{2}} \det(\mathbf{1} + \mathbf{B}) \left[\prod_{r=1}^{N} \int d\sigma_r\right] \times$$
$$\times \int_0^\infty dl l^{N-(p+3)/2} e^{-\mu_2 \int d^2\sigma \sqrt{g}} \prod_{i \neq j;\ i,j=1}^{N} e^{(\mathbf{k}_i \times \bar{\mathbf{k}}_j)\left(\sigma_{ij} - \frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)} \times$$

[8] An alternative treatment of the $l \to \infty$ limit focuses on the $O(1)$ term in the $e^{-\pi/l}$ expansion without performing the integration over the modular parameter. Upon applying momentum conservation, the integrals over the σ_i are performed, leaving the result in the form of an integral representation valid for generic kinematics [10]. The method could also be applied here.

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$$\times \left\{ \frac{2^4}{\pi} e^{-\frac{1}{2}\pi l \sigma_{ij}^2} \times \left(3 \ \sinh(\frac{1}{2}\pi l \sigma_{ij}) - \frac{1}{2}\sinh^2(\frac{1}{2}\pi l \sigma_{ij}) \right) \right\}, \tag{4.11}$$

where $\mathbf{p}$ is the net forward momentum transfer, understood to be absorbed by a massive closed string state. Notice that the world-sheet cosmological constant term regulates the large l behavior of the integral, any divergences being absorbed in a renormalization of the string coupling constant. The result can be expressed in the form:

$$\begin{aligned} \mathcal{A}_P|_{\text{decoup.}} &= i\tilde{g}^N \delta(\sum_{i=1}^N \mathbf{p}_i - \mathbf{p}) \prod_{i,j=1}^N \delta(\mathbf{p}_i \cdot \mathbf{p}_j - 1)(\alpha')^{-\frac{p+1}{2}} \det(\mathbf{1}+\mathbf{B}) \times \\ &\times \left[\prod_{r=1}^N \int d\sigma_r\right] \prod_{i\neq j;\ i,j=1}^N e^{(\mathbf{k}_i \times \bar{\mathbf{k}}_j)\left(\sigma_{ij} - \frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)} P(\sigma_1, \sigma_2, \ldots, \sigma_N), \end{aligned} \tag{4.12}$$

where P is a polynomial function of the $\sigma_i, i = 1, \ldots, N$, of order $n_p = \frac{(p+1)}{2} - N$, which can be expressed in the form:

$$P(\sigma_1, \sigma_2, \ldots, \sigma_N) = \sum_{m_N=0}^{n_p} \cdots \sum_{m_2=0}^{n_p} \sum_{m_1=0}^{n_p} \left(a^{(1)}_{m_1 m_2 \cdots m_N} \sigma_N^{m_N} \cdots \sigma_2^{m_2} \right) \sigma_1^{m_1}. \tag{4.13}$$

Note that with the chosen ordering of sources on the boundary, we have $\sigma_{ij} > 0$ for every $i > j$, with the integration domains, $0 \le \sigma_1 \le \sigma_2$, $0 \le \sigma_2 \le \sigma_3, \ldots$, and $0 \le \sigma_N \le 1$. The nested set of integrals over the σ_i can be evaluated as follows.

It is convenient to define the momentum dependent phases

$$\phi_j \equiv - \sum_{i\neq j, i=1}^N (\mathbf{k}_i \times \bar{\mathbf{k}}_j) = \frac{i}{2} \sum_{i\neq j, i=1}^N \Theta^{\mu\nu} p^i_\mu p^j_\nu, \tag{4.14}$$

in terms of which the innermost integral in eq. (4.12) takes the form:

$$\begin{aligned} I_1 &= \int_0^{\sigma_2} d\sigma_1 e^{-\phi_1 \sigma_1 + R_1(\sigma_2, \ldots, \sigma_N)} \sum_{m_1=0}^{n_p} \left(a^{(1)}_{m_1 m_2 \cdots m_N} \sigma_N^{m_N} \cdots \sigma_2^{m_2} \right) \sigma_1^{m_1} \\ &= e^{-\phi_2 \sigma_2 + R_2(\sigma_3, \ldots, \sigma_N)} \sum_{m_1=0}^{n_p} \left(a^{(1)}_{m_1 m_2 \cdots m_N} \sigma_N^{m_N} \cdots \sigma_3^{m_3} \right) \sigma_2^{m_2} \gamma\left(m_1 + 1, \phi_1 \sigma_2\right), \end{aligned} \tag{4.15}$$

where γ is an incomplete gamma function, and the R_j are linear in the σ_i, $i > j$. Substituting its polynomial representation, we iterate the procedure above to perform the next-to-innermost integral over σ_2:

$$I_2 = \int_0^{\sigma_3} d\sigma_2 e^{R_2(\sigma_3, \ldots, \sigma_N)} \sum_{m_2=0}^{n_p} \sum_{m_1=0}^{n_p} \left(a^{(1)}_{m_1 m_2 \cdots m_N} \sigma_N^{m_N} \cdots \sigma_3^{m_3} \right) [(m_1)!] \times$$

$$\times \left\{ e^{-\phi_2\sigma_2}\sigma_2^{m_2} - e^{-(\phi_1+\phi_2)\sigma_2} \left(\sum_{r_1=0}^{m_1} \frac{\phi_1^{r_1}}{r_1!} \sigma_2^{r_1+m_2} \right) \right\}$$

$$= e^{R_2(\sigma_3,\ldots,\sigma_N)} \sum_{m_2=0}^{n_p} \sum_{m_1=0}^{n_p} \left(a^{(1)}_{m_1 m_2 \cdots m_N} \sigma_N^{m_N} \cdots \sigma_3^{m_3} \right) [(m_1)!] \times$$

$$\times \left\{ \gamma\left(m_2+1, \phi_2\sigma_3\right) - \left[\sum_{r_1=0}^{m_1} \frac{\phi_1^{r_1}}{r_1!} \gamma\left(r_1+m_2+1, (\phi_1+\phi_2)\sigma_3\right) \right] \right\}$$

$$= e^{R_3(\sigma_4,\ldots,\sigma_N)} \sum_{m_2=0}^{n_p} \sum_{m_1=0}^{n_p} \left(a^{(1)}_{m_1 m_2 \cdots m_N} \sigma_N^{m_N} \cdots \sigma_3^{m_3} \right) \left[\prod_{i=1,2} (m_i)! \right] \times$$

$$\times \Bigg\{ e^{-\phi_3\sigma_3}\sigma_3^{m_3} - e^{-(\phi_2+\phi_3)\sigma_3} \left(\sum_{r_2=0}^{m_2} \frac{\phi_2^{r_2}}{r_2!} \sigma_3^{r_2+m_3} \right) -$$

$$-e^{-\phi_3\sigma_3} \sum_{r_1=0}^{m_1} \frac{(r_1+m_2)!}{m_2!} \frac{\phi_1^{r_1}}{r_1!} \sigma_3^{m_3} - e^{-(\phi_1+\phi_2+\phi_3)\sigma_3} \times$$

$$\times \sum_{r_2=0}^{m_2} \sum_{r_1=0}^{m_1} \frac{(r_1+m_2)!}{m_2!} \frac{(\phi_1+\phi_2)^{r_1+m_2+r_2}}{(r_1+m_2+r_2)!} \sigma_3^{r_1+m_2+r_2+m_3} \Bigg\}, \tag{4.16}$$

continuing through to the Nth integral. The result takes the form:

$$I_N = \sum_{m_N=0}^{n_p} \cdots \sum_{m_1=0}^{n_p} a^{(1)}_{m_1 m_2 \cdots m_N} \{ q(\phi_i; i \le N) - \sum_{j=1}^{N} \sum_{r_j=0}^{m_j} e^{-\phi_j} q_j(\phi_i; i \le N) + \cdots$$

$$+ e^{-(\phi_1+\phi_2+\cdots+\phi_N)} \sum_{r_N=0}^{m_N} \cdots \sum_{r_1=0}^{m_1} q_{1\cdots N}(\phi_i; i \le N) \}$$

$$\equiv e^{-(\phi_1+\phi_2+\cdots+\phi_N)} Q(\phi_1, \ldots, \phi_N) \,, \tag{4.17}$$

where the q's are polynomial functions of the ϕ_i. For particular values of p, N, this procedure can be made explicit inclusive of numerical factors. Thus, in contrast to the field theory limit of the planar amplitudes of free string theory in zero background B field and ordinary commutative space, the noncommutative amplitude is normalized by a momentum dependent phase function. We have shown that the functional dependence of the external momentum dependent phase function normalizing the planar amplitude takes the form:

$$\mathcal{A}_P|_{\text{decoup.}} = i\tilde{g}^N \delta(\sum_{i=1}^{N} \mathbf{p}_i - \mathbf{p}) \prod_{i,j=1}^{N} \delta(\mathbf{p}_i \cdot \mathbf{p}_j - 1) (\alpha')^{-\frac{p+1}{2}} \det(\mathbf{1}+\mathbf{B}) \times$$

$$\times e^{-\frac{1}{4}\sum_{\mu,\nu=0}^{p} \sum_{j=1}^{N} \sum_{i\neq j; i=1}^{N} i\Theta^{\mu\nu} p_\mu^i p_\nu^j} \times$$

$$\times Q\left(\phi_1, \phi_2, \ldots, \phi_N\right), \tag{4.18}$$

where we have substituted for the phases, ϕ_i, $i = 1, \ldots, N$, defined in eq. (4.14). Note that this expression holds for a chosen ordering of the external momenta about the boundary. The delta functions impose momentum conservation and the on-shell condition for the external momenta.

It is now apparent that the momentum dependent phase function has a natural interpretation as a wavefunction renormalization. Note that the result is a function of the variables, $\sum_{i\neq j;i=1}^{N}(\mathbf{p}_i\times\mathbf{p}_j)$, for every $j=1,\ldots,N$. Thus, the jth vertex operator with momentum $\mathbf{p}_j$ is normalized by a B dependent phase function determined by $\mathbf{p}_j$, and antisymmetric bilinear products coupling each $\mathbf{p}_j$ to the total remnant loop momentum due to the other sources: $\mathbf{p}-\mathbf{p}_j=\sum_{i\neq j}\mathbf{p}_i$.

5. World-sheet duality and non-planar amplitudes

The non-planar amplitude, with N_1 vertex operators on one boundary of the annulus and $N-N_1$ vertex operators on the other, takes the form:

$$\begin{aligned}\mathcal{A}_{NP}(1,\ldots,M) = i\tilde{g}^N\delta\left(\sum_{a=1}^{N}\mathbf{p}_a\right)(\alpha')^{-(p+1)/2}\det(\mathbf{1}+\mathbf{B})\left[\prod_{a=1}^{N}\int d\sigma_a\right]\times\\ \times\int_0^\infty\frac{dl}{l}\frac{l^{N-(p+1)/2}}{\eta(\frac{il}{2})^{24}}\left[e^{-\sum_{i,j=1}^{N_1}\sum_{m=0}^{(p-1)/2}k_i^m\bar{k}_j^m G'_P(\sigma_i^a,\sigma_j^a)}\right]\times\\ \times\left[e^{-\sum_{k=1}^{N_1}\sum_{u=N_1+1}^{N}\sum_{m=0}^{(p-1)/2}(k_k^m\bar{k}_u^m+k_u^m\bar{k}_k^m)G'_{NP}(\sigma_k^a,\sigma_u^a)}\right]\times\\ \times\left[e^{-\sum_{r,t=N_1+1}^{N}\sum_{m=0}^{(p-1)/2}k_r^m\bar{k}_t^m G'_P(\sigma_r^a,\sigma_t^a)}\right].\end{aligned}\qquad(5.1)$$

The derivation mirrors the earlier treatment of the planar amplitude. The Greens function for a single complex scalar with N_1 sources on the $\sigma^2=0$ boundary and $\sum_{i=1}^{N_1}\mathbf{p}_i=\mathbf{p}$, can be expressed in the form:

$$\begin{aligned}\sum_{i,j=1}^{N_1}k_i\bar{k}_jG'_P(\sigma_i,\sigma_j) &= \sum_{i\neq j;\ i,j=1}^{N_1}k_i\bar{k}_jG'_P(\sigma_i,\sigma_j)+\sum_{i=1}^{N_1}k_i\bar{k}_iG'_P(\sigma_i,\sigma_i)\\ &= \sum_{i\neq j;\ i,j=1}^{N_1}k_i\bar{k}_jG'_P(\sigma_i,\sigma_j)+\sum_{i=1}^{N_1}k_i\bar{k}G'_P(\sigma_i,\sigma_i)-\\ &\quad-\sum_{i\neq j;\ i,j=1}^{N_1}k_i\bar{k}_jG'_P(\sigma_i,\sigma_i)\\ &= \frac{1}{2}\sum_{i\neq j;\ i,j=1}^{N_1}\Big[(k_i\bar{k}_j+k_j\bar{k}_i)G'^{(e)}_P(\sigma_i,\sigma_j)+\\ &\qquad\qquad+(k_i\bar{k}_j-k_j\bar{k}_i)G'^{(o)}_P(\sigma_i,\sigma_j)\Big]+\\ &\quad+\sum_{i=1}^{N_1}k_i\bar{k}G'_P(\sigma_i,\sigma_i)-\frac{1}{2}\sum_{i\neq j;i,j=1}^{N}(k_i\bar{k}_j+k_j\bar{k}_i)G'_P(\sigma_i,\sigma_i)\end{aligned}\qquad(5.2)$$

and likewise for sources on the $\sigma^2 = 1$ boundary, but with k replaced by $-k$. The terms dependent on the internal momentum can be rewritten as the unrestricted double sums:

$$\begin{aligned} &\sum_{i=1}^{N}\sum_{r=N_1+1}^{N} k_i(-\bar{k}_r)G'_P(\sigma_i,\sigma_i) \qquad \text{for} \quad \sigma_i^2=0 \\ -&\sum_{r=N_1+1}^{N}\sum_{j=N_1}^{N} k_r(+\bar{k}_j)G'_P(\sigma_r,\sigma_r) \qquad \text{for} \quad \sigma_r^2=1\,, \end{aligned} \tag{5.3}$$

combining neatly with the unrestricted double sums over propagators, $G'_{NP}(\sigma_i,\sigma_r)$, between sources on distinct boundaries. Substituting from eqs. (3.14) and (3.16) in eq. (3.6) gives the expression:

$$\begin{aligned} &\prod_{i=1}^{N_1}\prod_{r=N_1+1}^{N} e^{\frac{2\pi\alpha' b^2}{l(1+b^2)}(\mathbf{k}_i\bar{\mathbf{k}}_r+\mathbf{k}_r\bar{\mathbf{k}}_i)}\left[e^{-2\pi l\sigma_{ir}^2\mathbf{k}_i\cdot\bar{\mathbf{k}}_r}\left|\frac{\Theta_2(\frac{il}{2}\sigma_{ir},\frac{il}{2})}{\pi\eta^3(\frac{il}{2})}\right|^{4\mathbf{k}_i\cdot\bar{\mathbf{k}}_r}\right] \equiv \\ &\qquad\qquad \equiv \prod_{i=1}^{N_1}\prod_{r=N_1+1}^{N} f_2^2(l;\sigma_{ir})e^{2\frac{\mathbf{k}_i\circ\bar{\mathbf{k}}_r}{l}}\,, \end{aligned} \tag{5.4}$$

Here, $\mathbf{k} = \sum_{i=1}^{N_1}\mathbf{k}_i = -\sum_{r=N_1+1}^{N}\mathbf{k}_r$ is the momentum transfer between the boundaries. Notice the appearance in the nonplanar amplitude of the symmetric product defined in eq. (4.1), corresponding to the spacetime invariant [8, 12]:

$$\sum_{i=1}^{N_1}\sum_{r=N_1+1}^{N}\mathbf{k}_i\circ\bar{\mathbf{k}}_r = -\mathbf{k}\circ\bar{\mathbf{k}} = \frac{1}{8\pi\alpha'}\sum_{\mu,\nu=0}^{p}\Theta^{\mu\sigma}(\mathcal{M})_{\sigma\lambda}\Theta^{\lambda\nu}\; p_\mu^i p_\nu^j\,, \tag{5.5}$$

where we have used momentum conservation in obtaining the first equality.

The remaining terms in eq. (5.2) combine with the restricted double sums over propagators, G'_P, for sources on the same boundary as in the derivation of the planar amplitude. Substituting in eq. (3.6) gives the result:[9]

$$\begin{aligned} \mathcal{A}_{NP}(1,\ldots,N) &= i\tilde{g}^N\delta(\sum_{a=1}^{N}\mathbf{p}_a)(\alpha')^{-(p+1)/2}\det(\mathbf{1}+\mathbf{B})\times \\ &\times\left[\prod_{r=1}^{N}\int d\sigma_r\right]e^{-2\frac{\mathbf{k}\circ\bar{\mathbf{k}}}{l}}\int_0^\infty\frac{dl}{l}\frac{l^{N-\frac{p+1}{2}}}{\eta(\frac{il}{2})^{24}}\times \\ &\times\prod_{i\neq j;\; i,j=1}^{N_1}\left[e^{\mathbf{k}_i\times\bar{\mathbf{k}}_j\left(\sigma_{ij}-\frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)}f_1(l;\sigma_{ij})\right]\left[\prod_{k=1}^{N_1}\prod_{u=N_1+1}^{N}f_2^2(l;\sigma_{ku})\right]\times \\ &\times\left[\prod_{r\neq t;\; r,t=N_1+1}^{N}e^{-\mathbf{k}_r\times\bar{\mathbf{k}}_t\left(\sigma_{rt}-\frac{1}{2}\frac{\sigma_{rt}}{|\sigma_{rt}|}\right)}f_1(l;\sigma_{rt})\right]. \end{aligned} \tag{5.6}$$

[9] Expressions for the nonplanar amplitude have been derived in [15, 16, 19] using the background field method. A different approach, based in part on the algebra of open string vertex operators in a background field, is discussed in [18, section 2].

The internal momentum dependent phase factor is intriguing [12], but we emphasize once again that it is dimensionless. Momentum transfer between boundaries is of course best studied in the closed string channel, using world-sheet open-closed string duality. This will enable us to examine carefully the limit of vanishing momentum transfer [12].

5.1 Zero momentum transfer

From the perspective of the closed string channel, the massless limit of the planar amplitude described in the previous section can be interpreted as the limit of infinite momentum transfer: the intermediary closed string states are infinitely massive, decoupling from the massless fields in the open string sector. It is instructive to consider the opposite regime of zero momentum transfer, dominated by massless closed string exchange. We begin by performing a change of variables, $s = 2/l$, in eq. (5.6), thereby expressing the nonplanar amplitude in a form appropriate for the study of zero and finite momentum transfer processes:

$$\begin{aligned}
&\mathcal{A}_{NP}(1, \dots, N) = \\
&\quad = i2^{-n_p} \tilde{g}^N \delta\left(\sum_{a=1}^{N} \mathbf{p}_a\right) (\alpha')^{-\frac{p+1}{2}} \det(\mathbf{1} + \mathbf{B}) \times \\
&\quad \times \left[\prod_{r=1}^{N} \int d\sigma_r\right] e^{-s\mathbf{k}\circ\bar{\mathbf{k}}} \int_0^{\infty} ds \frac{s^{-N+\frac{p-1}{2}-12}}{\eta(is)^{24}} \times \\
&\quad \times \prod_{i\neq j;\; i,j=1}^{N_1} \left[e^{\mathbf{k}_i\times\bar{\mathbf{k}}_j\left(\sigma_{ij}-\frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|}\right)} \tilde{f}_1(s;\sigma_{ij})\right] \left[\prod_{k=1}^{N_1} \prod_{u=N_1+1}^{N} \tilde{f}_4^2(s;\sigma_{ku})\right] \times \\
&\quad \times \left[\prod_{r\neq t;\; r,t=N_1+1}^{N} e^{-\mathbf{k}_r\times\bar{\mathbf{k}}_t\left(\sigma_{rt}-\frac{1}{2}\frac{\sigma_{rt}}{|\sigma_{rt}|}\right)} \tilde{f}_1(s;\sigma_{rt})\right],
\end{aligned} \tag{5.7}$$

where the functions $\tilde{f}_a(s, \sigma_{ij})$ take the form:

$$\tilde{f}_a(s;\sigma_{ij}) \equiv \left|\frac{\Theta_a(\sigma_{ij}, is)}{\pi s \eta^3(is)}\right|^{2\mathbf{k}_i\cdot\bar{\mathbf{k}}_j}. \tag{5.8}$$

The Jacobi theta functions, and their modular transformations, can be found in [9]. Eq. (5.7) is an equivalent starting point for an analysis of the nonplanar amplitude.

The contribution from massless closed string modes is obtained by isolating the $O(q^0)$ term in an expansion of the integrand of eq. (5.6) in powers of $q = e^{-2\pi s}$. Thus, the field theory limit of the nonplanar amplitude with only massless closed string modes mediating momentum transfer takes the form:

$$\lim_{\mathbf{p}\to 0} \mathcal{A}_{NP} = i2^{-n_p} \tilde{g}^N \delta\left(\sum_{a=1}^{N} \mathbf{p}_a\right) (\alpha')^{-(p+1)/2} \det(\mathbf{1} + \mathbf{B}) \times$$

$$
\begin{aligned}
&\times \left[\prod_{r=1}^{N}\int d\sigma_r\right]\int_0^\infty ds s^{-N+(p-1)/2-12} \times \\
&\times \prod_{i\neq j;\ i,j=1}^{N_1}\left(e^{(\mathbf{k}_i\times\bar{\mathbf{k}}_j)(\sigma_{ij}-\frac{1}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|})}|\frac{2}{\pi s}\sin(\pi\sigma_{ij})|^{2\mathbf{k}_i\cdot\bar{\mathbf{k}}_j}\right)\times \\
&\times e^{-s(\mathbf{k}\circ\mathbf{k})}\prod_{k=1}^{N_1}\prod_{u=N_1+1}^{N}\left(|\frac{1}{\pi s}e^{\pi s/4}|^{4\mathbf{k}_k\cdot\bar{\mathbf{k}}_u}\right)\times \\
&\times \prod_{r\neq t;\ r,t=N_1+1}^{N}\left(e^{(\mathbf{k}_r\times\bar{\mathbf{k}}_t)(\sigma_{rt}-\frac{1}{2}\frac{\sigma_{rt}}{|\sigma_{rt}|})}\ |\frac{2}{\pi s}\sin(\pi\sigma_{rt})|^{2\mathbf{k}_r\cdot\bar{\mathbf{k}}_t}\right)\times \\
&\times \left\{24+8\mathbf{k}_i\cdot\bar{\mathbf{k}}_j\sin^2(\pi\sigma_{ij})+12\mathbf{k}_k\cdot\bar{\mathbf{k}}_u+8\mathbf{k}_r\cdot\bar{\mathbf{k}}_t\sin^2(\pi\sigma_{rt})\right\}, \quad (5.9)
\end{aligned}
$$

upto terms of $O(e^{-2\pi s})$. It is convenient to express this in the form:

$$
\begin{aligned}
\lim_{\mathbf{p}\to 0}\mathcal{A}_{NP} &= i\tilde{g}^N\delta\left(\sum_{a=1}^{N}\mathbf{p}_a\right)(\alpha')^{-(p+1)/2}\det(\mathbf{1}+\mathbf{B})\left[\prod_{r=1}^{N}\int d\sigma_r\right]\times \\
&\times \int_0^\infty ds s^{-N+(p-1)/2-12}e^{-s\left[\mathbf{k}\circ\bar{\mathbf{k}}+(\mathbf{k}\cdot\bar{\mathbf{k}})\pi\right]}\left(\frac{1}{s^2}\right)^{\nu_k}\times \\
&\times \left[\prod_{i\neq j;\ i,j=1}^{N_1}\prod_{k=1}^{N_1}\prod_{u=N_1+1}^{N}\prod_{r\neq t;\ r,t=N_1+1}^{N}F_0^{(\text{closed})}(\sigma_a,\mathbf{k}_a)\right], \qquad (5.10)
\end{aligned}
$$

$F_0^{(\text{closed})}$ is a polynomial function of the $\sin(\pi\sigma_{ab})$, $a,b=1,\ldots,N$, denoting the contribution to the amplitude from the massless states in the closed string sector, and $\nu_k=-\sum_{a=1}^{N}\mathbf{k}_a\cdot\bar{\mathbf{k}}_a$. We have changed integration variables to the closed string modular parameter, $s=2/l$, and the leading $O(e^{2\pi s})$ contribution from the closed string tachyon has been suppressed in writing eq. (5.10). The term of $O(1)$ corresponds to the exchange of massless closed string states.

Remarkably, the B dependent terms in the exponent combine to give the ordinary Lorentz invariant bilinear in momentum transfer with euclidean, spacelike, signature:

$$
\mathbf{k}\circ\bar{\mathbf{k}}+\pi(\mathbf{k}\cdot\bar{\mathbf{k}})=\frac{1}{2}\pi\alpha'\sum_{m=0}^{\frac{p-1}{2}}\left(p_{2m}p_{2m}+p_{2m+1}p_{2m+1}\right), \qquad (5.11)
$$

free of any dependence on the background fields!. Integrating over the closed string modular parameter gives the result:

$$
\begin{aligned}
&\mathcal{A}_{NP}|_{\mathbf{p}=0} = \\
&= i\tilde{g}^N(\alpha')^{-\frac{p+1}{2}}\det(\mathbf{1}+\mathbf{B})\delta(\sum_{a=1}^{N}\mathbf{p}_a)\Gamma(\nu)\left[\frac{2}{\pi\alpha'\delta^{\mu\lambda}p_\mu p_\lambda}\right]^\nu\times \\
&\times\left[\prod_{r=1}^{N}\int d\sigma_r\right]\left[\prod_{i\neq j;\ i,j=1}^{N_1}\prod_{k=1}^{N_1}\prod_{u=N_1+1}^{N}\prod_{r\neq t;\ r,t=N_1+1}^{N}F_0^{(\text{closed})}(\sigma_a,\mathbf{k}_a)\right], \quad (5.12)
\end{aligned}
$$

where we have defined $\nu = \frac{p+1}{2} - N - 12 - 2\nu_k$. Note that the corrections to this result from $O(e^{-2\pi s})$ terms, due to the exchange of massive closed string states, replaces the single term in the integrand of eq. (5.9) with a series:

$$\sum_{n=0}^{\infty} F_n^{(\mathrm{closed})}(\sigma_a; \mathbf{k}_a) e^{-s[\mathbf{k}\circ\bar{\mathbf{k}} + \pi(\mathbf{k}\cdot\bar{\mathbf{k}}) + 2n\pi]}, \tag{5.13}$$

where the nth term denotes the contribution from states in the nth mass level in the closed string spectrum. Note also that eq. (5.12) holds for *arbitrary* external momenta: no kinematic constraints, or on-shell conditions, have been imposed on external momenta when deriving the zero momentum transfer limit in the closed string channel.

The amplitude has a pole whenever the momentum transfer equals the mass of an on-shell state in the closed string spectrum: $\alpha' \delta^{\mu\nu} p_\mu p_\nu = -4n$, with n taking all integer values including 0.[10] In particular, masses and couplings in the closed string sector, which appear at the loop level in this theory, scale naturally with respect to the bare fundamental string tension. In contrast, as mentioned earlier, the masses of states in the open string sector scale naturally with the effective string tension. Thus, for on-shell states, respectively in the nth mass level, we have the familiar bosonic string mass relations:

$$\begin{aligned} m^2_{\mathrm{closed}} &= -g^{\mu\nu} p_\mu p_\nu = \frac{4}{\alpha'}(n-1) \\ m^2_{\mathrm{open}} &= -p_\mu \left(\mathcal{M}^{-1}\right)^{\mu\nu} p_\nu = \frac{1}{\alpha'_{\mathrm{eff.}}}(n-1), \end{aligned} \tag{5.14}$$

For convenience, we have replaced the flat spacetime metric with its more general form, $g^{\mu\nu}$, as explained in footnote 4. This implies, also, the replacement: $\det(\mathbf{1} + \mathbf{B}) \to \det(\mathbf{g} + \mathbf{B})$, in the definition of the effective tension given in eq. (2.20). We note that this identification gives a world-sheet interpretation of the open and closed string metrics introduced in [8] . Note that, for a non-flat closed string metric, the closed string masses would scale in units of a different "effective" string tension, namely, $\alpha' \det(g)$. We return to this point in the conclusions.

We close with an important comment on the relation to the unoriented string. The integral over the closed string modular parameter exhibits a zero momentum divergence due to a tadpole when the index ν takes values:

$$\nu - 1 \equiv \frac{p-1}{2} - 12 - N + 2\sum_{a=1}^{N} \mathbf{k}_a \cdot \bar{\mathbf{k}}_a = 0\,. \tag{5.15}$$

We recover the familiar tadpole both in the vacuum amplitude, and in any N point

[10] Reference [23] argues for the decoupling of the closed string sector in nonplanar amplitudes based on the absence of poles in the expression for the nonplanar amplitude given in [19, eq. (2.17)], for S-dualized open and closed string metrics. We find no evidence for this in eqs. (5.12) and (5.13). Notice that the open and closed string metrics have identical spacetime dimension.

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on-shell tachyon amplitude, in the special case of space-filling D25branes, namely, $p = 25$ for zero momentum transfer [9]. Note that the tadpoles occur precisely as in the case of zero B field, except that the coupling to the vacuum is scaled *up* in magnitude by the factor $\det(\mathbf{1}+\mathbf{B})$. It should be recalled that the tadpole in the vacuum amplitude for $p = 25$ can be cancelled by considering instead the unoriented bosonic string theory [9], with 2^{13} D25branes and low energy gauge group $\mathrm{SO}(2^{13})$. The analysis of B field dependence proceeds as given above, except that the orientation projection imposes a quantization condition on the values of the background B field. The effective string tension will be raised relative to $(\alpha')^{-1/2}$ as before. Inclusion of the nonorientable diagrams does not bring in any new features other than are necessary to ensure tadpole cancellation, which works exactly as in the case of the zero field limit. In particular, there are no additional consistency conditions hidden in the scattering amplitudes over what was found in the vacuum amplitude.

5.2 Comments on stretched strings and the UV-IR correspondence

The momentum transfer dependent term in the nonplanar amplitude has been given a “stretched string” interpretation in [19]. We begin by pointing out that an interpretation of this term as the classical action for a stretched *open* string fails due to a discrepancy in the dependence on the world-sheet metric. We will use the notation of [19], noting that the fiducial cylinder metric used in [19] coincides with our conventions: $g^{11} = 1/l^2$, $g^{22} = 1$, with $\sqrt{g} = l$. The action for a stretched open string satisfying the boundary conditions:

$$X^\mu(\sigma^1, 1) - X^\mu(\sigma^1, 0) = \Theta^{\mu\nu} p_\nu + \text{oscillators}, \qquad (\Delta x)^2 = -p_\mu \left(\Theta G \Theta\right)^{\mu\nu} p_\nu \,, \tag{5.16}$$

takes the form:

$$S_{\text{cl.}} \sim (\Delta x)^2\, l \,, \tag{5.17}$$

rather than the desired term which is proportional to $1/l$! But in fact, from our analysis above, this term combines in the closed string channel with a $\mathbf{p}\cdot\mathbf{p}$ dependent term in the amplitude to give the ordinary Lorentz invariant momentum bilinear, subsequently eliminated by an integration over $s = 1/l$. Thus, in the small momentum transfer regime, probed by the closed string channel, we do not find the notion of a stretched string interpretation appropriate.

On the other hand, in the regime of large to infinite momentum transfer, it may indeed be helpful to interpret the momentum transfer dependent term as the classical action of a stretched string as follows. Recall that this regime is dominated by short open strings, and long, massive, closed strings, with $l \to \infty$. Consider the classical solution:

$$\begin{aligned}\tilde{x}^\mu(\sigma^1, \sigma^2) &= \pi\alpha' \Theta^{\mu\nu} p_\nu \left(\sigma^1 - \frac{1}{2}\right) \\ \partial_1 \tilde{x}^\mu \partial_1 \tilde{x}_\mu &= \pi^2 \alpha'^2 \Theta^{\mu\sigma} \left(\mathcal{M}\right)_{\sigma\lambda} \Theta^{\lambda\nu} p_\mu p_\nu = 2\pi\alpha' \mathbf{k} \circ \bar{\mathbf{k}} \,.\end{aligned} \tag{5.18}$$

Note that with our conventions, the constant B field dependent spacetime metric, $\tilde{G}$, is dimensionless. Substituting in the world-sheet action, using the fiducial cylinder metric, gives the desired result:

$$
\begin{aligned}
S_{\mathrm{cl.}}[\tilde{z}; g] &\equiv \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \tilde{G}_{\mu\nu} g^{ab} \partial_a \tilde{x}^\mu \partial_b \tilde{x}^\nu \\
&= \frac{1}{2} \frac{\mathbf{k} \circ \bar{\mathbf{k}}}{l} ,
\end{aligned}
\tag{5.19}
$$

where we have used the fact that $\sqrt{g} g^{11} = 1/l$. The classical configuration is a "rigid" ring-shaped annulus of large radius, and vanishingly small width. Large momentum transfer in the presence of the background B field gives this annular world-sheet a rigidity due to the antisymmetric $B^{\mu\nu}$ coupling. We emphasize that while we find this helpful classical intuition, we find *no connection* between the stretched string and the infra-red behavior of the theory. Nevertheless, open-closed string world-sheet duality has enabled us to rewrite the "stretched string action" — representing a strongly UV phenomenon from the perspective of the noncommutative field theory limit of the open string sector, in terms of dual closed string variables, where it now contributes to the poles in the nonplanar amplitude for on-shell closed string states — a decidedly IR effect from the perspective of the Lorentz invariant, and B field independent, commutative field theory of massless closed string states. This is simply another manifestation of the well-established UV-IR correspondence found in open and closed string perturbation theory.

We end by noting some of the unusual properties of the wavefunctions in the presence of a constant B field. Recall that the orthogonality coefficients are not positive definite, as is apparent in eq. (2.11). A consequence is an anomalously large contribution to the path integral whenever the ratio $n_2^2/n_1^2 = \alpha^2$. Ordinarily, one expects the contributions from large eigenvalue eigenfunctions to be damped by the $\omega_{n_1 n_2}^{-1}$ factor in the exponent of eq. (3.4). This is no longer true, the exponent periodically returning to an anomalously large contribution. In terms of the $O(e^{-\pi l})$ expansion, we note that this behavior represents anomalously large contributions of massive open string modes at specific points in the moduli space. However, these are also the special values at which the (n_1, n_2) states become degenerate with the $(n_1, 0)$ states. So, while this semi-classical intuition is appealing, we should emphasize that in the computation of the Greens function we were able to demonstrate an almost complete cancellation of the contributions from these anomalous modes.

6. Conclusions

Being one-dimensional objects, strings couple naturally to an antisymmetric two-form tensor field. The quantized particle associated with an antisymmetric two-form potential field belongs in the same spacetime Lorentz multiplet in string theory as the familiar graviton. In addition, superstring theories contain additional higher p-form background potentials with significant consequences for the nonperturbative dynamics of String/M theory [9], and, one would hope, equally significant consequences for string/M theoretic cosmology and particle physics [24, 25]. It is an important challenge to bring to the study of antisymmetric higher p-form gauge fields the intuition and detailed understanding we have of the physics of Yang-Mills gauge fields. Understanding the perturbative dynamics of open and closed string theory in background two-form tensor fields is a significant step in this direction.

One of this motivations of this work was to understand more precisely the implications of the possible noncommutativity of spacetime at short distances for String/M theory. As mentioned at the outset, spacetimes with constant background two-form tensor fields are but one route to noncommutativity in String/M theory. Our work has clarified some of the features which distinguish perturbative open and closed string theory in a noncommutative embedding space: the finite renormalization of the bare fundamental string tension, and the finite wavefunction renormalizations of open string vertex operators. We have derived, in the simplified kinematics of forward scattering with infinite momentum transfer, the precise form of the external momentum dependent phase functions normalizing planar amplitudes demonstrating that they admit interpretation as a renormalization of the wavefunctions of open string vertex operators. Using open-closed string world-sheet duality to probe the regime of finite and zero momentum transfer, we have shown explicitly the existence of poles in the nonplanar amplitude when the momentum transfer equals the mass of an on-shell closed string state. Remarkably, the field theory limit of the massless states in the closed string sector, entering through loops of open string, is an ordinary, Lorentz invariant, commutative quantum field theory with masses and couplings scaling naturally in units of the fundamental string tension. This is in sharp contrast to the field theory limit of the open string sector — a noncommutative field theory, with mass scale set by the effective string tension, larger than the bare fundamental string tension, and vertex operator algebra and wavefunction renormalization derived from the star product. We should emphasize that many of our conclusions are a re-interpretation in world-sheet terms of the insights in [8].

There is a well-established UV-IR correspondence in open and closed string theory, whereby a would-be ultraviolet phenomenon is recast as an infrared pheneomenon in the dual string variables [9]. From the mass formulae for on-shell states in the open and closed string spectrum given in eq. (5.14), it becomes clear that the notion of "bare" and "effective" string scales are, as a consequence of world-sheet open-closed string duality, interchangeable. While it is the bare fundamental string scale that appears in the world-sheet action, the effective string scale is actually *higher* in energy, probing shorter distance scales. In the context of wilsonian renormalization in the presence of fixed background fields, it is therefore natural to think of the, lower, fundamental closed string scale as "derived". Conversely, if we think in terms of the moduli space with varying background fields, we can interpret the generation of an effective string scale *higher* than the closed string scale, as a *finite* renormalization effect.[11]

We have found no evidence in the course of our analysis to support the notion that renormalizability in the wilsonian sense is exotic in these spacetimes. The main distinction here between conclusions drawn from noncommutative field theory [12] and from string theory lies in the application of open-closed string world-sheet duality. We emphasize that, while a bilinear product identical in form to the $\mathbf{p} \circ \mathbf{p}$ invariant found in [12] is indeed present in the open string channel of the nonplanar amplitude, it is *dimensionless*. Using world-sheet open-closed string duality, we have further shown that the dependence on internal momentum transfer in the closed string channel is purely through the ordinary Lorentz invariant momentum bilinear with flat (euclidean signature) metric. The nonplanar amplitude factorizes on poles at the usual on-shell values for states in the closed string spectrum. Thus, we find no puzzles either in the ultraviolet or the infrared regimes. Renormalization of the open string coupling proceeds precisely as in a commutative spacetime, taking the UV cutoff to infinity while absorbing the divergences in the propagator in the short distance limit into a renormalization of the bare open string coupling [13]. The low momentum transfer regime of the nonplanar amplitudes of open string theory is instead studied by dualizing to the closed string channel. All potential divergences appear in the guise of an IR effect, determined by the dynamics of the zero momentum massless modes in the closed string sector. Other than the well-known divergences associated with the tadpole in the vacuum amplitude for the D25brane, we find no new divergences in the closed string channel of the nonplanar amplitude. The $\mathbf{p} \to 0$ limit is benign, quite independent of the procedure by which the ultraviolet cutoff on the open string loop momentum is taken to infinity in establishing coupling constant renormalization.

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A. Computation of the Greens function

The infinite series appearing in the expression for the Greens function given in eq. (3.6) can be simplified by using the residue theorem and applying a Sommerfeld

[11] Finite, only because we have, of course, assumed finite values of the background fields such that our world-sheet computations are well-founded.

Watson transform. An elementary introduction adequate to follow the discussion can be found in the texts [26]. The basic idea is to use the residue theorem to get rid of the infinite sums over the n_1 index.The first step is to rewrite the sum over n_1 as an integral:

$$\sum_{n_1=-\infty}^{\infty} f(n_1) = \int_C dz\ f(z)\frac{\cot(\pi z)}{i} - S_1\,. \tag{A.1}$$

In the above equation, the contour C runs above the real axis, from $\infty + i\epsilon$ to $\infty - i\epsilon$. The terms in the integrand are such that they go to zero as z goes to either $+i\infty$ or $-i\infty$, and one can perform a contour integration by closing the contour in the upper or lower half-plane, respectively. The part of the integral that is evaluated by closing in the lower half-plane picks up residues from poles at all integers n, which come from the factor $\mathrm{Cot}(\pi z)$, and produce the infinite sum. In addition, there are a finite number of other poles in the integrand whose residues contribute to the integral an extra term S_1 , which we subtract. The next step is to rewrite the integrand in such a way that any term with poles at integer n goes to zero as z goes to $+i\infty$, and we can close the contour in the upper half-plane. The integral thus does not pick up any infinite sum over n, but rather has a finite number of terms, which we denote by S_2:

$$\int_C dz\ f(z)\frac{\cot(\pi z)}{i} = S_2\,. \tag{A.2}$$

After which we can combine the two equations to get

$$\sum_{n_1=-\infty}^{\infty} f(n_1) = S_2 - S_1\,. \tag{A.3}$$

We now illustrate this process with specific examples.

To begin with, consider the series:

$$\begin{aligned}\sum_{n_1=-\infty}^{\infty} f(n_1) &\equiv \sum_{n_1=-\infty}^{\infty} \frac{e^{-2\pi i n_1 \sigma^1_-}}{n_1^2 + n_2^2 l^2/4} \\ &= \sum_{n_1=-\infty}^{\infty} \frac{\cos(2\pi i n_1 \sigma^1_-)}{n_1^2 + n_2^2 l^2/4}\,.\end{aligned} \tag{A.4}$$

We can rewrite this as an integral as follows:

$$\begin{aligned}\int_C dz \frac{\cos(2\pi z \sigma^1_-)}{z^2 + n_2^2 l^2/4}\frac{\cot(\pi z)}{i} &= \frac{1}{2i}\int_C dz \frac{\cos(2\pi z \sigma^1_-)}{z^2 + n_2^2 l^2/4}\frac{e^{i\pi z} + e^{-i\pi z}}{\sin(\pi z)} \\ &= \frac{2\pi i}{2i}(-1)\left[\frac{\cos(2\pi\sigma^1_- i l n_2/2)}{2 i l n_2/2}\frac{e^{-\pi l n_2/2}}{\sin(i\pi l n_2/2)}\right] + \\ &\quad + \frac{2\pi i}{2i}\left[\frac{\cos(-2\pi\sigma^1_- i l n_2/2)}{-2 i l n_2/2}\frac{e^{-\pi l n_2/2}}{\sin(-i\pi l n_2/2)}\right] + \\ &\quad + \frac{2\pi i}{2i}\left[\frac{1}{\pi}\sum_{n_1=-\infty}^{\infty}\frac{\cos(2\pi i n_1 \sigma^1_-)}{n_1^2 + n_2^2 l^2/4}\right].\end{aligned} \tag{A.5}$$

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In the second line, the term in the integrand proportional to $e^{i\pi z}$ has been evaluated using the residue theorem by closing the contour in the upper half-plane, picking up a residue at the pole $z = in_2 l/2$ and an overall minus sign from clockwise integration. The term proportional to $e^{-i\pi z}$ has been closed in the lower-half-plane, picking up residues at all integers $z = n_1$ as well as at $z = -in_2 l/2$. The residues of the imaginary poles cancel in this expression, leaving only the sum we are interested in evaluating, so $S_1 = 0$ in this case. To proceed, we rearrange the integrand using

$$\begin{aligned} \frac{\cot(\pi z)}{i} &= \frac{2e^{i\pi z}}{e^{i\pi z} - e^{-i\pi z}} - 1 \\ &= \frac{e^{i\pi z} + e^{-i\pi z}}{2i \quad \sin(\pi z)}\,, \end{aligned} \tag{A.6}$$

after which the integral becomes

$$\int_C dz \frac{\cos(2\pi z\sigma_-^1)}{n_1^2 + n_2^2 l^2/4}\left(\frac{2e^{i\pi z}}{e^{i\pi z} - e^{-i\pi z}} - 1\right). \tag{A.7}$$

We can now evaluate the first term by closing in the upper half-plane, where there is only a simple pole at $z = iln_2/2$, and we no longer have a contribution from the poles at $z = n$. To evaluate the second term, we need to split the integrand by writing $\cos(2\pi z\sigma_-^1) = (1/2)e^{i2\pi z\sigma_-^1} + (1/2)e^{-i2\pi z\sigma_-^1}$. We then close one term in the upper half-plane and the other in the lower half-plane, where the terms go to zero. After doing this, one arrives at

$$\begin{aligned} \sum_{n_1=-\infty}^{\infty} \frac{e^{-2\pi i n_1 \sigma_-^1}}{n_1^2 + n_2^2 l^2/4} &= S_2 \\ &= 2\pi i\Bigg[-\frac{\cosh(\pi n_2 l\sigma_-^1)}{in_2 l}\frac{2e^{-\pi l n_2/2}}{e^{-\pi l n_2/2} - e^{\pi l n_2/2}} - \\ &\quad -\frac{1}{2}\left(-\frac{e^{-\pi n_2 l|\sigma_-^1|}}{in_2 l} + \frac{e^{-\pi n_2 l|\sigma_-^1|}}{-in_2 l}\right)\Bigg] \\ &= \frac{4\pi}{n_2 l}\left[\cosh(\pi n_2 l\sigma_-^1)\frac{e^{-\pi l n_2}}{1 - e^{-\pi l n_2}} + \frac{1}{2}e^{-\pi n_2 l|\sigma_-^1|}\right]. \end{aligned} \tag{A.8}$$

Another sum we will need to evaluate is of the form

$$\sum_{n_1=-\infty}^{\infty} g(n_1) \equiv \sum_{n_1=-\infty}^{\infty} \frac{e^{-2\pi i n_1\sigma_-^1}}{n_2^2 - n_1^2\alpha^2}\,. \tag{A.9}$$

We use the same trick to transform this sum, although we now have poles along the real axis that contribute to S_1. Consider the integral

$$\frac{1}{2i}\int_C dz \frac{\cos(2\pi z\sigma_-^1)}{n_2^2 - z^2\alpha^2}\frac{e^{i\pi z} + e^{-i\pi z}}{\sin(\pi z)} = \frac{2\pi i}{2i}\left[-(0) + \sum_{n_1=-\infty}^{\infty}\frac{\cos(2\pi n_1\sigma_-^1)}{n_2^2 - n_1^2\alpha^2}\frac{1}{\pi}\right] +$$

$$
\begin{aligned}
&+\frac{2\pi i}{2i}\left[\frac{\cos(2\pi\sigma_-^1 n_2/\alpha)}{-2\alpha n_2}\frac{e^{-i\pi n_2/\alpha}}{\sin(\pi n_2/\alpha)}\right]+\\
&+\frac{2\pi i}{2i}\left[\frac{\cos(-2\pi\sigma_-^1 n_2/\alpha)}{2\alpha n_2}\frac{e^{i\pi n_2/\alpha}}{\sin(-\pi n_2/\alpha)}\right]\\
=&\sum_{n_1=-\infty}^{\infty}\frac{\cos(2\pi n_1\sigma_-^1)}{n_2^2-n_1^2\alpha^2}+S_1\,. \qquad \text{(A.10)}
\end{aligned}
$$

For the first term, we've closed the contour in the upper half-plane where the integrand is analytic, giving a vanishing contribution to the integral. The second term is closed in the lower half plane and gives contributions from poles at integer values $z = n_1$ as well as $z = \pm n_2/\alpha$. After rewriting the integrand using eq. (A.6) we arrive at

$$
\begin{aligned}
\int_C dz\frac{\cos(2\pi z\sigma_-^1)}{n_2^2-z^2\alpha^2}\left(\frac{2e^{i\pi z}}{e^{i\pi z}-e^{-i\pi z}}-1\right)&=S_2\\
&=2\pi i\left[-(0)-\frac{1}{2}\left(\frac{e^{-i2\pi n_2|\sigma_-^1|/\alpha}}{-2\alpha n_2}+\frac{e^{i2\pi n_2|\sigma_-^1|/\alpha}}{2\alpha n_2}\right)\right]. \qquad \text{(A.11)}
\end{aligned}
$$

We then arrive at the following expression:

$$
\begin{aligned}
\sum_{n_1=-\infty}^{\infty}\frac{e^{-2\pi i n_1\sigma_-^1}}{n_2^2-n_1^2\alpha^2}&=S_2-S_1\\
&=\frac{\pi}{\alpha n_2}\left[\cos(2\pi n_2|\sigma_-^1|/\alpha)\cot(\pi n_2/\alpha)+\sin(2\pi n_2|\sigma_-^1|/\alpha)\right]\\
&=\frac{\pi}{\alpha n_2}\frac{\cos[(2|\sigma_-^1|-1)\pi n_2/\alpha]}{\sin(\pi n_2/\alpha)}\,. \qquad \text{(A.12)}
\end{aligned}
$$

Using this method, one can also evaluate

$$
\begin{aligned}
\sum_{n_1=-\infty}^{\infty}n_1 f(n_1)&=\frac{\sigma_-^1}{|\sigma_-^1|}2\pi i\left(\sinh(\pi n_2 l|\sigma_-^1|)\frac{e^{-\pi l n_2}}{1-e^{-\pi l n_2}}-\frac{1}{2}e^{-\pi n_2 l|\sigma_-^1|}\right)\\
\sum_{n_1=-\infty}^{\infty}n_1 g(n_1)&=-\frac{\sigma_-^1}{|\sigma_-^1|}\frac{\pi i}{\alpha^2}\frac{\sin[(2|\sigma_-^1|-1)\pi n_2/\alpha]}{\sin(\pi n_2/\alpha)}\,. \qquad \text{(A.13)}
\end{aligned}
$$

This leaves us with the sum over $n_2 = 0$ modes. Defining $\sigma \equiv \sigma_-^1 + \frac{b}{l}\sigma_+^2$, we wish to evaluate the sum,

$$
\sum_{n_1=-\infty}^{\infty\prime}\frac{e^{i\pi n_1(\alpha-2\sigma)}}{n_1\sin(\pi n_1\alpha)}\,. \qquad \text{(A.14)}
$$

In order to exhibit the cancellation between contributions from this term and the trigonometric terms obtained above, we resum using a Sommerfeld-Watson transform. We therefore consider the integral:

$$
\begin{aligned}
I &= \frac{1}{2i}\int_C \frac{1}{z}\frac{e^{i\pi z(\alpha-2\sigma)}}{\sin(\pi z\alpha)}\frac{e^{i\pi z}+e^{-i\pi z}}{\sin(\pi z)} \\
&= \frac{2\pi i}{2i}\left[-(0)+\sum_{n_1=-\infty}^{\infty\prime}\frac{e^{i\pi n_1(\alpha-2\sigma)}}{n_1\sin(\pi n_1\alpha)}\frac{1}{\pi}+\sum_{n_1=-\infty}^{\infty\prime}\frac{e^{-i\pi n_1/\alpha(2\sigma+1)}}{n_1\sin(\pi n_1/\alpha)}\frac{1}{\pi}\right]+ \\
&\quad +\frac{2\pi i}{2i}\left[\frac{1}{\alpha}\left(\frac{1}{6}-\frac{\alpha^2}{3}-\frac{1}{2}(2\sigma+1)^2+\alpha(2\sigma+1)\right)\right].
\end{aligned}
\tag{A.15}
$$

We have closed the $e^{i\pi z}$ term in the upper half plane where it is analytic and makes no contribution to the integral. The $e^{-i\pi z}$ term has been closed in the lower half plane, picking up simple poles at $z=n$, $z=n/\alpha$ for integer n, and a triple pole at $z=0$. The integrand can be rewritten as follows:

$$
I=\int_C \frac{1}{z}\frac{e^{i\pi z(\alpha-2\sigma)}}{\sin(\pi z\alpha)}\left(\frac{2e^{i\pi z}}{e^{i\pi z}-e^{-i\pi z}}-1\right). \tag{A.16}
$$

The first term can always be closed in the upper half-plane, giving a zero contribution to the integral. For the second term, we close in the upper half-plane for $\sigma_-^1<0$, in the lower half-plane for $\sigma_-^1>0$, giving

$$
I=\begin{cases} 0\,, & \sigma_-^1<0 \\ \sum_{n_1=-\infty}^{\infty\prime}\frac{-2i}{n_1}e^{-2i\pi n_1\sigma/\alpha}+\frac{2\pi}{\alpha}(\alpha-2\sigma)\,, & \sigma_-^1>0 \end{cases}. \tag{A.17}
$$

Combining these equations leads to the expression

$$
\begin{aligned}
\sum_{n_1=-\infty}^{\infty\prime}\frac{e^{i\pi n_1(\alpha-2\sigma)}}{n_1\sin(\pi n_1\alpha)} &= -2\sum_{n_1=1}^{\infty\prime}\frac{\cos(\pi n_1/\alpha(2\sigma-\frac{\sigma_-^1}{|\sigma_-^1|}))}{n_1\sin(\pi n_1/\alpha)}- \\
&\quad -\frac{\pi}{\alpha}\left(\frac{1}{6}-\frac{\alpha^2}{3}-\frac{1}{2}(2\sigma-\frac{\sigma_-^1}{|\sigma_-^1|})^2+\alpha(2\sigma-\frac{\sigma_-^1}{|\sigma_-^1|})\right).
\end{aligned}
\tag{A.18}
$$

From eq. (3.6), the Greens function for a single complex scalar in constant background B field can be written as:

$$
\begin{aligned}
G(\sigma_i^a,\sigma_j^a) &= \frac{2\alpha' l}{\pi(1+b^2)}\sum_{n_1=-\infty}^{\infty}e^{-2\pi i n_1\sigma_-^1}\sum_{n_2=1}^{\infty}\left(\frac{1}{n_1^2+n_2^2l^2/4}+\frac{\alpha^2}{n_2^2-n_1^2\alpha^2}\right)\times \\
&\quad \times\left[\cos(\pi n_2\sigma_+^2)-i\frac{\alpha n_1}{n_2}\sin(\pi n_2\sigma_+^2)\right]+ \\
&\quad +\frac{2\alpha' l}{\pi}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=1}^{\infty}\frac{e^{-2\pi i n_1\sigma_-^1}}{n_1^2+n_2^2l^2/4}\sin(\pi n_2\sigma_i^2)\sin(\pi n_2\sigma_j^2)+ \\
&\quad +\sum_{n_1=-\infty}^{\infty\prime}\left[\frac{2\alpha' b}{1+b^2}e^{-2\pi i n_1\sigma}\frac{1}{n_1}\frac{e^{i\pi n_1\alpha}}{\sin(\pi n_1\alpha)}\right].
\end{aligned}
\tag{A.19}
$$

where the prime in the last term denotes exclusion of the $n_1 = 0$ mode. Rewriting the sum over n_1 using the Sommerfeld Watson transform gives:

$$
\begin{aligned}
&G(\sigma_i^a, \sigma_j^a) = \\
&= \frac{8\alpha'}{1+b^2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\cosh(\pi n l |\sigma_-^1|) \frac{e^{-\pi n l}}{1 - e^{-\pi n l}} + \frac{1}{2} e^{-\pi n l |\sigma_-^1|} \right) \cos(\pi n \sigma_+^2) + \\
&+ \frac{8\alpha' b}{1+b^2} \frac{\sigma_-^1}{|\sigma_-^1|} \sum_{n=1}^{\infty} \frac{1}{n} \left(\sinh(\pi n l |\sigma_-^1|) \frac{e^{-\pi n l}}{1 - e^{-\pi n l}} - \frac{1}{2} e^{-\pi n l |\sigma_-^1|} \right) \sin(\pi n \sigma_+^2) + \\
&+ 8\alpha' \sum_{n=1}^{\infty} \frac{1}{n} \left(\cosh(\pi n l |\sigma_-^1|) \frac{e^{-\pi n l}}{1 - e^{-\pi n l}} + \frac{1}{2} e^{-\pi n l |\sigma_-^1|} \right) \sin(\pi n \sigma_i^2) \sin(\pi n \sigma_j^2) + \\
&+ \frac{4\alpha' b}{1+b^2} \sum_{n=1}^{\infty} \frac{\cos((2|\sigma_-^1| - 1)\pi n/\alpha)}{n \sin(\pi n/\alpha)} \cos(\pi n \sigma_+^2) - \\
&- \frac{\sigma_-^1}{|\sigma_-^1|} \frac{4\alpha' b}{1+b^2} \sum_{n=1}^{\infty} \frac{\sin((2|\sigma_-^1| - 1)\pi n/\alpha)}{n \sin(\pi n/\alpha)} \sin(\pi n \sigma_+^2) - \\
&- \frac{4\alpha' b}{1+b^2} \sum_{n=1}^{\infty} \frac{\cos\left(\left(2\sigma - \frac{\sigma_-^1}{|\sigma_-^1|}\right) \pi n/\alpha\right)}{n \sin(\pi n/\alpha)} - \\
&- \frac{\pi \alpha' l}{1+b^2} \left(\frac{1}{6} - \frac{\alpha^2}{3} - \frac{1}{2} \left(2\sigma - \frac{\sigma_-^1}{|\sigma_-^1|} \right)^2 + \alpha (2\sigma - \frac{\sigma_-^1}{|\sigma_-^1|}) \right). \qquad \text{(A.20)}
\end{aligned}
$$

Notice that the last three sums, with $\sin(\pi n/\alpha)$ in the denominator, cancel each other out. We can then rewrite this expression in terms of theta functions by using the Taylor expansions

$$
\frac{e^{-\pi n l}}{1 - e^{-\pi n l}} = \sum_{m=1}^{\infty} e^{-\pi l n m}, \quad \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x), \qquad \text{(A.21)}
$$

and by defining

$$
\nu_{\pm} = \frac{il}{2} \sigma_-^1 + \frac{1}{2} \sigma_{\pm}^2 \quad , \tau = \frac{il}{2}. \qquad \text{(A.22)}
$$

After doing this, we arrive finally at our result:

$$
\begin{aligned}
G'(\sigma_i^a, \sigma_j^a) = &- \frac{2\alpha'}{1+b^2} \ln \frac{|\Theta_1(\nu_+, \tau)|^2}{|\eta(\tau)|^2} - \\
&- \frac{2i\alpha' b}{1+b^2} \ln \frac{\Theta_1(\nu_+, \tau)}{\Theta_1^\star(\nu_+, \tau)} - \alpha' \ln \left| \frac{\Theta_1(\nu_-, \tau)}{\Theta_1(\nu_+, \tau)} \right|^2 + \\
&+ \frac{2\pi \alpha' l}{1+b^2} \left[\sigma^2 - \frac{2b\sigma}{l} + \frac{2b^2}{3l^2} \right]. \qquad \text{(A.23)}
\end{aligned}
$$

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